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Origins of State Capacity (Besley and Persson)

Derivations of Formulas

Deriving (4), p. 6, Besley and Persson

From (2) and (3)

Income of a high-return individual in group J

$$\left[r_H + p_S^J (r_H - r_L) \right] w^J$$

Income of a low-return individual in group J

$$r_L w^J$$

σ^J: share of group J agents with high returns

⇒ Average income of an individual in group J:

$$\sigma^J \left[r_H + p_S^J (r_H - r_L) \right] w^J + (1 - \sigma^J) r_L w^J$$

$$= \sigma^J \left[r_H + p_S^J (r_H - r_L) - r_L \right] w^J + r_L w^J$$

$$= \left[\sigma^J (1 + p_S^J)(r_H - r_L) + r_L \right] \cdot w^J$$

Proof of Propositions (1) and (2) (p. 8 and p. 9)

The Incumbent Government has the following objective function

$$f(t_s^J, t_s^K, p_s^J, p_s^K, G_s)$$

$$= \alpha_s G_s + \bar{\rho}(1-t_s^J)\beta^J Y(p_s^J, \sigma^J, w^J) + \underline{\rho}(1-t_s^K)\beta^K Y(p_s^K, \sigma^K, w^K)$$

subject to the following constraints

$$p_s^J, p_s^K \le \Pi_s, \quad t_s^J, t_s^K \le T_s$$

$$\sum_J t_1^J \beta^J Y(p_s^J, \sigma^J, w^J) = G_1 + L(\Pi_2 - \Pi_1) + F(T_2 - T_1)$$

$$\sum_J t_2^J \beta^J Y(p_s^J, \sigma^J, w^J) = G_2$$

Substitute $G_1 = \sum_J t_1^J \beta^J Y(p_s^J, \sigma^J, w^J) - L(\Pi_2 - \Pi_1) - F(T_2 - T_1)$ and G_2

into the objective function, we obtain:

$$f(t_s^J, t_s^K, p_s^J, p_s^K)$$

$$= \alpha_s \cdot \left[t_s^J \beta^J Y(p_s^J, \sigma^J, w^J) + t_s^K \beta^K Y(p_s^K, \sigma^K, w^K) - \overbrace{L(\Pi_2 - \Pi_1) + F(T_2 - T_1)}^{\text{if } s=1} \right]$$

$$+ \bar{\rho}(1-t_s^J)\beta^J Y(p_s^J, \sigma^J, w^J) + \underline{\rho}(1-t_s^K)\beta^K Y(p_s^K, \sigma^K, w^K)$$

$$= (\alpha_s - \bar{\rho}) t_s^J \beta^J Y(p_s^J, \sigma^J, w^J) + (\alpha_s - \underline{\rho}) t_s^K \beta^K Y(p_s^K, \sigma^K, w^K)$$

$$+ \bar{\rho}\beta^J Y(p_s^J, \sigma^J, w^J) + \underline{\rho}\beta^K Y(p_s^K, \sigma^K, w^K) - \underbrace{\alpha_s \left[L(\Pi_2 - \Pi_1) + F(T_2 - T_1) \right]}_{\text{if } s=1}$$

with $Y(p_s^J, \sigma^J, w^J) = \left[\sigma^J (1 + p_s^J)(r_H - r_L) + r_L \right] w^J$

We maximize this (the last) expression of $f(t_s^J, t_s^K, p_s^J, p_s^K)$

subject to the following constraints $p_s^J \le \Pi_s$, $p_s^K \le \Pi_s$, $t_s^K \le \tau_s$, $t_s^J \le \tau_s$

Derived from the budget constraints $\Leftarrow$
$$\begin{cases} -\sum_J t_1^J \beta^J Y(p_s^J, \sigma^J, w^J) \le L(\Pi_2 - \Pi_1) - F(\tau_2 - \tau_1) \\ -\sum_J t_2^J \beta^J Y(p_s^J, \sigma^J, w^J) \le 0 \end{cases}$$

Period 1 :

$$L_1 = f(t_1^J, t_1^K, p_1^J, p_1^K) - \lambda_1(p_1^J - \Pi_1) - \lambda_2(p_1^K - \Pi_1) - \lambda_3(t_1^J - \tau_1) - \lambda_4(t_1^K - \tau_1)$$
$$- \lambda_5\left[-\sum t_1^J \beta^J Y(p_1^J, \sigma^J, w^J) + L(\Pi_2 - \Pi_1) + F(\tau_2 - \tau_1)\right]$$

FOCs

$$\frac{\partial L_1}{\partial p_1^J} = \frac{\partial f}{\partial p_1^J} - \lambda_1 + \lambda_5 \cdot t_1^J \beta^J \frac{\partial Y(p_1^J, \sigma^J, w^J)}{\partial p_1^J}$$

$$= \left[(\alpha_1 - \bar{\rho}) t_1^J + \bar{\rho}\right] \cdot \beta^J \cdot \frac{\partial Y(p_1^J, \sigma^J, w^J)}{\partial p_1^J} - \lambda_1 + \lambda_5 t_1^J \beta^J \frac{\partial Y(p_1^J, \sigma^J, w^J)}{\partial p_1^J}$$

$$= \left[(\alpha_1 - \bar{\rho} + \lambda_5) t_1^J + \bar{\rho}\right] \cdot \beta^J \cdot \sigma^J w^J (r_H - r_L) - \lambda_1$$
$$= \left[(\alpha_1 + \lambda_5) t_1^J + (1 - t_1^J)\bar{\rho}\right] \beta^J \sigma^J w^J (r_H - r_L) - \lambda_1 = 0$$

$$\frac{\partial L_1}{\partial p_1^K} = \left[(\alpha_1 - \underline{\rho} + \lambda_5) t_1^K + \underline{\rho}\right] \cdot \beta^K \sigma^K w^K (r_H - r_L) - \lambda_2$$
$$= \left[(\alpha_1 + \lambda_5) t_1^K + (1 - t_1^K)\underline{\rho}\right] \cdot \beta^K \sigma^K w^K (r_H - r_L) - \lambda_2 = 0$$

$$\frac{\partial L_1}{\partial t_1^J} = (\alpha_1 - \bar{\rho}) \beta^J Y(p_1^J, \sigma^J, w^J) - \lambda_3 + \lambda_5 \cdot \beta^J Y(p_1^J, \sigma^J, w^J)$$
$$(\alpha_1 - \bar{\rho} + \lambda_5) \cdot \beta^J \left[\sigma^J (1 + p_1^J)(r_H - r_L) + r_L\right] w^J - \lambda_3 = 0$$

$$\frac{\partial L_1}{\partial t_1^K} = (\alpha_1 - \underline{\rho} + \lambda_5) \beta^K \left[\sigma^K (1 + p_1^K)(r_H - r_L) + r_L\right] w^K - \lambda_4 = 0$$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0$

$$\lambda_1 (p_1^J - \pi_1) = 0$$

$$\lambda_2 (p_1^K - \pi_1) = 0$$

$$\lambda_3 (t_1^J - \tau_1) = 0$$

$$\lambda_4 (t_1^K - \tau_1) = 0$$

$$\lambda_5 \left[-\sum t_1^J \beta^J Y(p_1^J, \sigma^J, w^J) + L(\pi_2 - \pi_1) + F(\tau_2 - \tau_1) \right] = 0$$

Since the first term of $\frac{\partial L_1}{\partial p_1^J} > 0 \Rightarrow \lambda_1 > 0 \Rightarrow p_1^J - \pi_1 = 0 \Rightarrow \boxed{p_1^J = \pi_1}$

the first term of $\frac{\partial L_1}{\partial p_1^K} > 0 \Rightarrow \lambda_2 > 0 \Rightarrow p_1^K - \pi_1 = 0 \Rightarrow \boxed{p_1^K = \pi_1}$

⊛ <u>If $\bar{\rho} = \underline{\rho} = 1$</u>

- If $\alpha_1 > 1$ $\Rightarrow$ the first term of $\frac{\partial L_1}{\partial t_1^J} > 0 \Rightarrow \lambda_3 > 0 \Rightarrow \boxed{t_1^J = \tau_1}$

  the first term of $\frac{\partial L_1}{\partial t_1^K} > 0 \Rightarrow \lambda_4 > 0 \Rightarrow \boxed{t_1^K = \tau_1}$

  $\Rightarrow \boxed{G_1 = \tau_1 \sum_J \beta^J Y(\pi_1, \sigma^J, w^J) - L(\pi_2 - \pi_1) - F(\tau_2 - \tau_1)}$

- If $\alpha_1 < 1$ $\Rightarrow$ From the expression $\frac{\partial L_1}{\partial t_1^J}$, and $\lambda_3 \geq 0 \Rightarrow \alpha_1 - \bar{\rho} + \lambda_5 \geq 0$

  $\Rightarrow \alpha_5 > 0$

  $\Rightarrow \sum_J t_1^J \beta^J Y(p_1^J, \sigma^J, w^J) = L(\pi_2 - \pi_1) + F(\tau_2 - \tau_1) \Rightarrow \boxed{G_1 = 0}$

  $\Rightarrow$ Model predicts incumbent governments' indifference to any (t_1^J, t_2^J) satisfying the above equation. Authors "assumed, without loss of generality", $t_1^J = t_2^J = \hat{t}_1$ (p.9)

- If $\alpha_1 = 1$ $\Rightarrow$ May have $\lambda_5 = \lambda_3 = \lambda_4 = 0 \Rightarrow t_1^J \leq \tau_1, t_1^K \leq \tau_1,$

  $\sum_J t_1^J \beta^J Y(p_5^J, \sigma^J, w^J) \geq L(\pi_2 - \pi_1) + F(\tau_2 - \tau_1)$

⊛ <u>If $\bar{\rho} > \underline{\rho}$:</u>

- If $\alpha_1 > \bar{\rho}$ $\Rightarrow$ the 1st term of $\frac{\partial L_1}{\partial t_1^J} > 0 \Rightarrow \lambda_3 > 0 \Rightarrow \boxed{t_1^J = \tau_1}$

  the 1st term of $\frac{\partial L_1}{\partial t_1^K} > 0 \Rightarrow \lambda_4 > 0 \Rightarrow \boxed{t_1^K = \tau_1}$

$$\Rightarrow \boxed{G_1 = \tau_1 \sum_J \beta^J Y(\pi_1, \sigma^J, w^J) - L(\pi_2 - \pi_1) - F(\tau_2 - \tau_1)}$$

• If $\alpha_1 < \bar{\rho}$: From the expression of $\frac{\partial L_1}{\partial t_1^J}$, and from $\lambda_3 \geq 0$

$$\Rightarrow \alpha_1 - \bar{\rho} + \lambda_5 \geq 0 \quad \Rightarrow \lambda_5 > 0$$

$$\Rightarrow \sum_J t_1^J \beta^J Y(\pi_1, \sigma^J, w^J) = L(\pi_2 - \pi_1) + F(\tau_2 - \tau_1)$$

$$\Rightarrow \boxed{G_1 = 0}$$

Also, since $\alpha_1 - \bar{\rho} + \lambda_5 \geq 0$

and $\alpha_1 - \underline{\rho} + \lambda_5 > \alpha_1 - \bar{\rho} + \lambda_5$ (because $\bar{\rho} > \underline{\rho}$)

$$\Rightarrow \alpha_1 - \underline{\rho} + \lambda_5 > 0$$

$\Rightarrow$ the first term of $\frac{\partial L_1}{\partial t_1^K} > 0 \Rightarrow \lambda_4 > 0$

$$\Rightarrow \boxed{t_1^K = \tau_1}$$

$$\Rightarrow \boxed{t_1^J = \frac{L(\pi_2 - \pi_1) + F(\tau_2 - \tau_1) - \tau_1 \beta^K Y(\pi_1, \sigma^K, w^K)}{\beta^J Y(\pi_1, \sigma^J, w^J)}}$$

• If $\alpha_1 = \bar{\rho}$ $\Rightarrow$ the first term of $\frac{\partial L_1}{\partial t_1^K} > 0 \Rightarrow \lambda_4 > 0 \Rightarrow \boxed{t_1^K = \tau_1}$

May have $\lambda_3 = \lambda_5 = 0$,

$G_1 \geq 0$

$$\sum_J t_1^J \beta^J Y(\pi_1, \sigma^J, w^J) \geq L(\pi_2 - \pi_1) + F(\tau_2 - \tau_1)$$

and $t_1^J \leq \tau_1$

Period 2

Since the optimization problem of period 1 differs from that of period 2 only in the budget constraint

in period 1: $-\sum_J t_1^J \beta^J Y(p_1^J, \sigma^J, w^J) \leq -L(\Pi_2 - \Pi_1) - F(\tau_2 - \tau_1)$

in period 2: $-\sum_J t_2^J \beta^J Y(p_2^J, \sigma^J, w^J) \leq 0$

(p. 3 of this document)

Solutions of the period 2 optimization problem are (for comparison with period 1 solution, one could refer to p. 4-5 of this document)

* $p_2^J = p_2^K = \Pi_2$

* If $\bar{\rho} = \underline{\rho} = 1$

  - If $\alpha_2 > 1 \Rightarrow t_2^J = \tau_2$; $t_2^K = \tau_2$; $G_2 = \tau_2 \sum_J \beta^J Y(\Pi_2, \sigma^J, w^J)$

  - If $\alpha_2 < 1 \Rightarrow \sum_J t_2^J \beta^J Y(p_2^J, \sigma^J, w^J) = 0$, $G_2 = 0$

  - If $\alpha_2 = 1$, again, may have $\lambda_5 = \lambda_3 = \lambda_4 = 0$
    $\sum_J t_2^J \beta^J Y(\Pi_2, \sigma^J, w^J) \geq 0$, $t_2^J \leq \tau_2$, $t_2^K \leq \tau_2$

* If $\bar{\rho} > \underline{\rho}$

  - If $\alpha_2 > \bar{\rho}$, $t_2^J = \tau_2$; $t_2^K = \tau_2$; $G_2 = \tau_2 \cdot \sum_J \beta^J Y(\Pi_2, \sigma^J, w^J)$

  - If $\alpha_2 < \bar{\rho}$, $G_2 = 0$; $t_2^K = \tau_2$; $t_2^J = -\dfrac{\tau_2 \cdot \beta^K Y(\Pi_2, \sigma^K, w^K)}{\beta^J Y(\Pi_2, \sigma^J, w^J)}$

  - If $\alpha_2 = \bar{\rho}$, $t_2^K = \tau_2$, may have $\lambda_3 = \lambda_5 = 0$

    $\Rightarrow \sum_J t_2^J \beta^J Y(\Pi_2, \sigma^J, w^J) \geq 0$

    $t_2^J \leq \tau_2$

Derivation of (9) from (15) and (16) [(15), (16) appear in Appendix, p. 21, 22]

Note that (15), (16) are derived from (8), and

that in (16), the authors made a small typo error,

the last term is written as $\tau_2(\underline{\rho}-\bar{\rho})\beta^J Y(\Pi_s, \sigma^K, w^K)$

it should be $\tau_2(\underline{\rho}-\bar{\rho})\beta^J Y(\Pi_s, \sigma^J, w^J)$

Deriving (9):

- Note that the 1st term in (15) = the 1st term in (16). Let them = M

  the 2nd term in (15) = the 2nd term in (16). Let them = N

  Let the 3rd term in (15) = P

  and the 3rd term in (16) = Q

- Then:

$$W^J(\tau_2, \Pi_2) = \gamma^J E\{W_J^J(\alpha_2, \tau_2, \Pi_2)\} + (1-\gamma^J)E\{W_K^J(\alpha_2, \tau_2, \Pi_2)\}$$

$$= \gamma^J[M + (1-H(\bar{\rho})).E(N) + H(\bar{\rho}).P]$$

$$+ (1-\gamma^J).[M + (1-H(\bar{\rho})).E(N) + H(\bar{\rho}).Q]$$

$$= M + [1-H(\bar{\rho})].E(N) + H(\bar{\rho}).[\gamma^J.P + (1-\gamma^J).Q]$$

$$= \bar{\rho}\beta^J Y(\Pi_2, \sigma^J, w^J) + \underline{\rho}\beta^K Y(\Pi_2, \sigma^K, w^K)$$

$$+ (1-H(\bar{\rho})).\tau_2.E\underbrace{\left[(\alpha_2-\bar{\rho}).\beta^J.Y(\Pi_2, \sigma^J, w^J) + (\alpha_2-\underline{\rho})\beta^K Y(\Pi_2, \sigma^K, w^K)\middle|\alpha_2 > \bar{\rho}\right]}_{(N)}$$

$$+ H(\bar{\rho}).\left[\underbrace{\gamma^J.\tau_2(\bar{\rho}-\underline{\rho})\beta^K Y(\Pi_2, \sigma^K, w^K)}_{(P)} + \underbrace{(1-\gamma^J).\tau_2.(\underline{\rho}-\bar{\rho})\beta^J Y(\Pi_2, \sigma^J, w^J)}_{(Q)}\right]$$

Let $J = \beta^J Y(\Pi_2, \sigma^J, w^J)$ and $K = \beta^K Y(\Pi_2, \sigma^K, w^K)$, then

$$w^J_{(\tau_2,\pi_2)} = \bar\rho J + \underline\rho K + (1-H(\bar\rho))\cdot \tau_2 \cdot E\left[(\alpha_2-\bar\rho)J + (\alpha_2-\underline\rho)K \mid \alpha_2 \geq \bar\rho\right]$$
$$+ H(\bar\rho)\cdot\tau_2\cdot\left[\gamma^J(\bar\rho-\underline\rho)K + (1-\gamma^J)(\underline\rho-\bar\rho)J\right]$$

$$= \bar\rho J + \underline\rho K + \tau_2\cdot(1-H(\bar\rho))\cdot E\left[\alpha_2(J+K) - (\bar\rho J+\underline\rho K)\mid\alpha_2\geq\bar\rho\right]$$
$$+\tau_2\cdot H(\bar\rho)\cdot\left[\gamma^J\bar\rho K - \gamma^J\underline\rho K + (1-\gamma^J)\underline\rho J - (1-\gamma^J)\bar\rho J\right]$$

$$=\bar\rho J+\underline\rho K+\tau_2(1-H(\bar\rho))\left[E(\alpha_2)\cdot(J+K) - (\bar\rho J+\underline\rho K)\mid\alpha_2\geq\bar\rho\right]$$
$$+\tau_2\cdot H(\bar\rho)\cdot\left[-(\bar\rho J+\underline\rho K)+\gamma^J\bar\rho K + (1-\gamma^J)\underline\rho K + (1-\gamma^J)\underline\rho J + \gamma^J\bar\rho J\right]$$

$$=\bar\rho J+\underline\rho K+\tau_2\cdot(1-H(\bar\rho))\cdot\left[E(\alpha_2)\cdot(J+K) - (\bar\rho J+\underline\rho K)\mid\alpha_2\geq\bar\rho\right]$$
$$+\tau_2\cdot H(\bar\rho)\cdot\left[-(\bar\rho J+\underline\rho K) + \left(\gamma^J\bar\rho + (1-\gamma^J)\underline\rho\right)\cdot(J+K)\right]$$

$$=\left[\bar\rho J+\underline\rho K\right]\cdot\left[1-\tau_2(1-H(\bar\rho)) - \tau_2\cdot H(\bar\rho)\right] + \tau_2\cdot\left[1-H(\bar\rho)\right]\cdot E(\alpha_2\mid\alpha_2\geq\bar\rho)\cdot(J+K)$$
$$+\tau_2\cdot H(\bar\rho)\cdot\left(\gamma^J\bar\rho + (1-\gamma^J)\underline\rho\right)\cdot(J+K)$$

$$=\left[\bar\rho J+\underline\rho K\right]\cdot\left[1-\tau_2\right] + \tau_2\cdot\left[(1-H(\bar\rho))\cdot E(\alpha_2\mid\alpha_2\geq\bar\rho) + H(\bar\rho)\cdot\left(\gamma^J\bar\rho+(1-\gamma^J)\underline\rho\right)\right]\cdot$$
$$\cdot\left[J+K\right]$$

$$=(1-\tau_2)\cdot\left[\bar\rho\,\beta^J Y(\pi_2,\sigma^J,w^J) + \underline\rho\,\beta^K Y(\pi_2,\sigma^K,w^K)\right]$$
$$+\tau_2\cdot\left[(1-H(\bar\rho))\cdot E(\alpha_2\mid\alpha_2\geq\bar\rho) + H(\bar\rho)\cdot\left(\gamma^J\bar\rho+(1-\gamma^J)\underline\rho\right)\right]$$
$$\cdot\left[\beta^J Y(\pi_2,\sigma^J,w^J) + \beta^K Y(\pi_2,\sigma^K,w^K)\right]$$

Deriving (12) (p. 11, Besley & Persson)

Note: $Y(\pi_2, \sigma^J, w^J) = \left[\sigma^J (1+\pi_2)(r_H - r_L) + r_L\right] \cdot w^J$

$$W^J_{(\tau_2, \pi_2)} = (1-\tau_2)\cdot\left[\bar{\rho}\beta^J Y(\pi_2, \sigma^J, w^J) + \underline{\rho}\beta^K Y(\pi_2, \sigma^K, w^K)\right]$$
$$+ \tau_2 \cdot \left[(1-H(\bar{\rho}))\cdot E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho})\cdot(\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho})\right]$$
$$\cdot \left[\beta^J Y(\pi_2, \sigma^J, w^J) + \beta^K Y(\pi_2, \sigma^K, w^K)\right]$$

$$\frac{\partial W^J(\tau_2, \pi_2)}{\partial \pi_2} = (1-\tau_2)\cdot\left[\bar{\rho}\beta^J\sigma^J(r_H - r_L)w^J + \underline{\rho}\beta^K\sigma^K(r_H - r_L)w^K\right]$$
$$+ \tau_2 \cdot \left[(1-H(\bar{\rho}))\cdot E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho})\cdot(\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho})\right]$$
$$\cdot \left[\beta^J\sigma^J(r_H - r_L)w^J + \beta^K\sigma^K(r_H - r_L)w^K\right]$$

$$= (1-\tau_2)\left[\bar{\rho}\omega^J\Omega(r_H - r_L) + \underline{\rho}\omega^K\Omega(r_H - r_L)\right]$$
$$+ \tau_2 \cdot \left[(1-H(\bar{\rho}))E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho})(\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho}\right]\cdot(r_H - r_L)(\omega^J\Omega + \omega^K\Omega)$$

($\omega^J, \omega^K, \Omega$ defined under (10) p. 11)

$$= (1-\tau_2)(r_H - r_L)\Omega(\bar{\rho}\omega^J + \underline{\rho}\omega^K)$$
$$+ \tau_2 \cdot \left[(1-H(\bar{\rho}))E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho})(\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho}\right](r_H - r_L)\Omega$$

(since $\omega^J + \omega^K = 1$)

$$= (r_H - r_L)\Omega(\bar{\rho}\omega^J + \underline{\rho}\omega^K)$$
$$+ (r_H - r_L)\Omega\,\tau_2 \cdot \left[(1-H(\bar{\rho}))\cdot E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho})(\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho}) - \bar{\rho}\omega^J - \underline{\rho}\omega^K\right]$$

Since $\bar{\rho}\omega^J + \underline{\rho}\omega^K = \bar{\rho}\omega^J + \underline{\rho}(1-\omega^J) = \underline{\rho} + (\bar{\rho} - \underline{\rho})\omega^J = \rho^J$

defined in formula (11), p. 11, Besley & Persson

$$\frac{\partial w^J(\tau_2, \Pi_2)}{\partial \Pi_2} = (r_H - r_L)\,\Omega\,\rho^J$$

$$+ (r_H - r_L)\,\Omega\,\tau_2 \cdot \left[(1 - H(\bar{\rho})) \cdot E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho})(\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho}) - \underline{\rho} - (\bar{\rho} - \underline{\rho}) w^J\right]$$

In the 2nd term,

$$(1 - H(\bar{\rho})) \cdot E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho}) \cdot (\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho}) - \underline{\rho} - (\bar{\rho} - \underline{\rho}) w^J$$

$$= (1 - H(\bar{\rho})) \cdot \left[E(\alpha_2 | \alpha_2 \geq \bar{\rho}) - \underline{\rho} - (\bar{\rho} - \underline{\rho}) w^J\right] + H(\bar{\rho}) \left[\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho} - \underline{\rho} - (\bar{\rho} - \underline{\rho}) w^J\right]$$

$$= \left[1 - H(\bar{\rho})\right] \cdot \left[E(\alpha_2 | \alpha_2 \geq \bar{\rho}) - \underline{\rho} - (\bar{\rho} - \underline{\rho}) w^J\right] + H(\bar{\rho}) \cdot \left[\gamma^J (\bar{\rho} - \underline{\rho}) - w^J (\bar{\rho} - \underline{\rho})\right]$$

$$= \left[1 - H(\bar{\rho})\right] \cdot \left[E(\alpha_2 | \alpha_2 \geq \bar{\rho}) - \underline{\rho} - (\bar{\rho} - \underline{\rho}) w^J\right] + H(\bar{\rho}) \cdot (\gamma^J - w^J)(\bar{\rho} - \underline{\rho})$$

Therefore, $$\frac{\partial w^J(\tau_2, \Pi_2)}{\partial \Pi_2} = (r_H - r_L)\,\Omega\,(\rho^J + \tau_2 \lambda_2^J)$$

where $$\lambda_2^J = \left[1 - H(\bar{\rho})\right] \left[E(\alpha_2 | \alpha_2 \geq \bar{\rho}) - \underline{\rho} - (\bar{\rho} - \underline{\rho}) w^J\right] + H(\bar{\rho}) (\gamma^J - w^J)(\bar{\rho} - \underline{\rho})$$

Deriving (13) (p. 11, Besley & Persson)

Note : $Y(\Pi_2, \sigma^J, w^J) = \left[\sigma^J(1+\Pi_2)(r_H - r_L) + r_L\right] \cdot w^J$

$$W^J(\tau_2, \Pi_2) = (1-\tau_2) \cdot \left[\bar{\rho}\beta^J Y(\Pi_2, \sigma^J, w^J) + \underline{\rho}\beta^K Y(\Pi_2, \sigma^K, w^K)\right]$$
$$+ \tau_2 \cdot \left[(1-H(\bar{\rho})) \cdot E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho}) \cdot (\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho})\right]$$
$$\cdot \left[\beta^J Y(\Pi_2, \sigma^J, w^J) + \beta^K Y(\Pi_2, \sigma^K, w^K)\right]$$

$$\frac{\partial W^J(\tau_2, \Pi_2)}{\partial \tau_2} = -\bar{\rho}\beta^J\left[\sigma^J(1+\Pi_2)(r_H - r_L) + r_L\right]w^J - \underline{\rho}\beta^K\left[\sigma^K(1+\Pi_2)(r_H - r_L) + r_L\right]w^K$$
$$+ \left[(1-H(\bar{\rho})) \cdot E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho}) \cdot (\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho})\right]$$
$$\cdot \left[\beta^J\left[\sigma^J(1+\Pi_2)(r_H - r_L) + r_L\right]w^J + \beta^K\left[\sigma^K(1+\Pi_2)(r_H - r_L) + r_L\right]w^K\right]$$

Note from page 10 of this document (2nd page of the derivation of formula (12)) that

$$\lambda_2^J = \left[1-H(\bar{\rho})\right] \cdot \left[E(\alpha_2 | \alpha_2 \geq \bar{\rho}) - \underline{\rho} - (\bar{\rho} - \underline{\rho})\omega^J\right] + H(\bar{\rho}) \cdot (\gamma^J - \omega^J)(\bar{\rho} - \underline{\rho})$$
$$= \left[1-H(\bar{\rho})\right] \cdot E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho}) \cdot (\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho}) - \left[\underline{\rho} + (\bar{\rho} - \underline{\rho})\omega^J\right]$$

(This appears below "in the 2nd term" also on p. 10 of this document.)

Since $\underline{\rho} + (\bar{\rho} - \underline{\rho})\omega^J = \bar{\rho}\omega^J + \underline{\rho}(1-\omega^J) = \bar{\rho}\omega^J + \underline{\rho}\omega^K$

($\omega^J + \omega^K = 1$ as explained on p. 11 of Besley & Persson's paper)

We have

$$\lambda_2^J = \left[1-H(\bar{\rho})\right].\ E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho})(\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho}) - \left[\bar{\rho} w^J + \underline{\rho} w^K\right]$$

Thus

$$\frac{\partial W^J(\tau_2, \pi_2)}{\partial \tau_2} = \left[(1 - H(\bar{\rho}))\, E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho}).\,(\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho})\right]$$

$$.\left[\beta^J\left[\sigma^J(1+\pi_2)(r_H - r_L) + r_L\right]w^J + \beta^K\left[\sigma^K(1+\pi_2)(r_H - r_L) + r_L\right]w^K\right]$$

$$- \bar{\rho}\beta^J\left[\sigma^J(1+\pi_2)(r_H - r_L) + r_L\right]w^J - \underline{\rho}\beta^K\left[\sigma^K(1+\pi_2)(r_H - r_L) + r_L\right]w^K$$

$$= \left[(1 - H(\bar{\rho})).\,E(\alpha_2 | \alpha_2 \geq \bar{\rho}) + H(\bar{\rho}).(\gamma^J \bar{\rho} + (1-\gamma^J)\underline{\rho}) - (\bar{\rho} w^J + \underline{\rho} w^K)\right]$$

$$.\left[w^J \Omega(1+\pi_2)(r_H - r_L) + r_L \beta^J w^J + w^K \Omega(1+\pi_2)(r_H - r_L) + r_L \beta^K w^K\right]$$

$$+\left[\bar{\rho} w^J + \underline{\rho} w^K\right].\left[w^J \Omega(1+\pi_2)(r_H - r_L) + r_L \beta^J w^J + w^K \Omega(1+\pi_2)(r_H - r_L) + r_L \beta^K w^K\right]$$

$$- \bar{\rho}\left[w^J \Omega(1+\pi_2)(r_H - r_L) + \beta^J w^J r_L\right] - \underline{\rho}\left[w^K \Omega(1+\pi_2)(r_H - r_L) + \beta^K w^K r_L\right]$$

(Since $\beta^J \sigma^J w^J = w^J \Omega$ and $\beta^K \sigma^K w^K = w^K \Omega$, p. 11, Besley & Persson)

$$= \lambda_2^J.\left[\Omega(1+\pi_2)(r_H - r_L) + r_L(\beta^J w^J + \beta^K w^K)\right]$$

$$+\left[\bar{\rho} w^J + \underline{\rho} w^K\right]\left[\Omega(1+\pi_2)(r_H - r_L) + r_L(\beta^J w^J + \beta^K w^K)\right]$$

$$- \bar{\rho}\left[w^J \Omega(1+\pi_2)(r_H - r_L) + \beta^J w^J r_L\right] - \underline{\rho}\left[w^K \Omega(1+\pi_2)(r_H - r_L) + \beta^K w^K r_L\right]$$

(Since $w^J + w^K = 1$, p. 11, Besley and Persson)

$$= \lambda_2^J \left[\Omega (1+\pi_2)(r_H - r_L) + r_L (\beta^J w^J + \beta^K w^K) \right]$$

$$+ r_L \cdot \left\{ \left[\bar{\rho} w^J + \underline{\rho} w^K \right] \cdot \left[\beta^J w^J + \beta^K w^K \right] - \bar{\rho} \beta^J w^J - \underline{\rho} \beta^K w^K \right\}$$

(Since $\bar{\rho} w^J \Omega (1+\pi_2)(r_H - r_L)$ is canceled out from the 2nd & 3rd term

$\underline{\rho} w^K \Omega (1+\pi_2)(r_H - r_L)$ is canceled out from the 2nd & 4th term

in the last expression on the previous page, p. 12)

$$= \lambda_2^J \left[\Omega (1+\pi_2)(r_H - r_L) + r_L (\beta^J w^J + \beta^K w^K) \right]$$

$$+ r_L \left\{ \left[\bar{\rho} w^J + \underline{\rho} w^K \right] \cdot \left[\beta^J w^J + \beta^K w^K \right] - \bar{\rho} \left[w^J + w^K \right] \beta^J w^J - \underline{\rho} \left[w^J + w^K \right] \beta^K w^K \right\}$$

(Since $w^J + w^K = 1$, p. 11, Besley and Persson)

$$= \lambda_2^J \left[\Omega (1+\pi_2)(r_H - r_L) + r_L (\beta^J w^J + \beta^K w^K) \right]$$

$$+ r_L \cdot \left\{ \bar{\rho} \left[w^J \beta^K w^K - w^K \beta^J w^J \right] + \underline{\rho} \left[w^K \beta^J w^J - w^J \beta^K w^K \right] \right\}$$

$$= \lambda_2^J \left[\Omega (1+\pi_2)(r_H - r_L) + r_L (\beta^J w^J + \beta^K w^K) \right]$$

$$+ r_L \cdot \left[\bar{\rho} - \underline{\rho} \right] \cdot \left[w^J \beta^K w^K - w^K \beta^J w^J \right]$$

$$= \lambda_2^J \left[\Omega (1+\pi_2)(r_H - r_L) + r_L (\beta^J w^J + \beta^K w^K) \right]$$

$$+ r_L \left[\bar{\rho} - \underline{\rho} \right] \cdot \frac{\beta^J w^J \beta^K w^K (\sigma^J - \sigma^K)}{\Omega}$$

(Since $w^J = \dfrac{\beta^J w^J \sigma^J}{\Omega}$ and $w^K = \dfrac{\beta^K w^K \sigma^K}{\Omega}$)