Wealth Distribution in Finite Life

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Abstract

I study the wealth distribution in a finite life model. Keeping the Pareto tail as in Benhabib and Bisin (2008), I try to incorporate three more realistic mechanisms to generate wealth inequality: (1) the luck within the life time, (2) the death rate from the real data, and (3) the life time income profile from the real data (including retirement).

1. Introduction

There are several prominent reasons that cause the wealth inequality: the different life length, the different labor income due to age, the stochastic rate of return on wealth, and inheritance. In this paper I try to incorporate these factors in a model to replicate the wealth inequality that the real data display. And decompose the contribution of these factors to the wealth inequality.

2. Model

The agent lives from 0 to T. $Y(t)$ is the labor income depending on age. Agents have portfolio selection problem between a risky asset and a riskless asset.

Consumer's problem

$$J(W,t) = \max_{C,\omega} E_t \int_t^T \frac{C(s)^{1-\gamma}}{1-\gamma} e^{-\theta(s-t)} ds + \chi \frac{[(1-\zeta)W(T)]^{1-\gamma}}{1-\gamma} e^{-\theta(T-t)}$$

$$s.t. \quad dW(s) = [rW(s) + (\alpha - r)\omega(s)W(s) - C(s) + Y(s)]ds + \sigma\omega(s)W(s)dB(s)$$

Define

$$b(t) = \int_t^T Y(s)e^{-r(s-t)}ds$$

We know that

$$db(t) = [-Y(t) + rb(t)]dt$$

Proposition 1. *The agent's policy functions are*

$$C(t) = a(t)^{-\frac{1}{\gamma}}[W(t) + b(t)]$$

$$\omega(t)W(t) = \frac{\alpha - r}{\gamma\sigma^2}[W(t) + b(t)]$$

where

$$a(t) = \begin{pmatrix} (\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} \\ +\left((\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} + \frac{1}{\frac{1-\gamma}{\gamma}(r+\frac{1}{2}\frac{(\alpha-r)^2}{\gamma\sigma^2}) - \frac{\theta}{\gamma}}\right)\left(\exp([\frac{1-\gamma}{\gamma}(r + \frac{1}{2}\frac{(\alpha-r)^2}{\gamma\sigma^2}) - \frac{\theta}{\gamma}](T-t)) - 1\right) \end{pmatrix}^{\gamma}$$

And

$$\begin{aligned} d(W(t) + b(t)) &= \left(r + \frac{(\alpha - r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right)(W(t) + b(t))dt \\ &\quad + \frac{\alpha - r}{\gamma\sigma}(W(t) + b(t))dB(t) \end{aligned}$$

Let

$$X(t) = W(t) + b(t)$$

Thus

$$dX(t) = \left(r + \frac{(\alpha - r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right)X(t)dt + \frac{\alpha - r}{\gamma\sigma}X(t)dB(t)$$

We can solve

$$X(t) = \left(\frac{a(t)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[(\frac{r - \theta}{\gamma} + \frac{(\alpha - r)^2}{2\gamma\sigma^2})t + \frac{\alpha - r}{\gamma\sigma}B(t)]X(0)$$

Note that

$$
\begin{aligned}
W(T) &= X(T) \\
&= \left(\frac{a(T)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2})T + \frac{\alpha-r}{\gamma\sigma}B(T)]X(0) \\
&= (\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} a(0)^{-\frac{1}{\gamma}} \exp[(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2})T + \frac{\alpha-r}{\gamma\sigma}B(T)]X(0)
\end{aligned}
$$

2.1. Intergenerational connection

Now let T, $2T$, $3T$,$\cdots$, nT,$\cdots$ be the born time of generation 1, 2, 3,$\cdots$, n,$\cdots$. Let

$$X_1 = X(T), X_2 = X(2T), X_3 = X(3T), \cdots, X_n = X(nT), \cdots$$

Thus

$$
\begin{aligned}
X_{n+1} &= X((n+1)T) \\
&= (1-\zeta)W((n+1)T) + b(0) \\
&= \left(\frac{\chi(1-\zeta)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2})T + \frac{\alpha-r}{\gamma\sigma}B(T)]X(nT) + b(0) \\
&= \left(\frac{\chi(1-\zeta)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2})T + \frac{\alpha-r}{\gamma\sigma}B(T)]X_n + b(0)
\end{aligned}
$$

Let

$$\rho_{n+1} = \left(\frac{\chi(1-\zeta)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2})T + \frac{\alpha-r}{\gamma\sigma}B(T)]$$

Note that ρ_{n+1} is lognormally disributed.

Thus

$$X_{n+1} = \rho_{n+1}X_n + b(0)$$

Thus the result of Sornette could be applied here.

3. Pareto tail

First I prove that the bequest distribution has a Pareto upper tail. Then by Reed (2006), I claim that the wealth distribution has an asymptotic Pareto upper tail.

3.1. Bequest distribution

By Sornette, the bequest follows a distribution with a Pareto upper tail, if there exists a μ such that

$$E\rho_{n+1}^{\mu} = 1$$

Note that ρ_{n+1} is log-normally distributed. Thus

$$E\rho_{n+1}^{\mu} = \left(\frac{\chi(1-\zeta)}{a(0)}\right)^{\frac{\mu}{\gamma}} \exp[\mu(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2})T + \frac{1}{2}\mu^2\frac{(\alpha-r)^2}{\gamma^2\sigma^2}T] = 1$$

We have

$$\mu = \gamma \left(\frac{\frac{1}{T}\log\left(\frac{a(0)}{\chi(1-\zeta)}\right) + \theta - r}{\frac{(\alpha-r)^2}{2\sigma^2}} - 1 \right)$$

4. Simulation reults

Using the labor income estimated from Consumer Expenditure Survey (CEX), I simulate the wealth distribution of the economy.

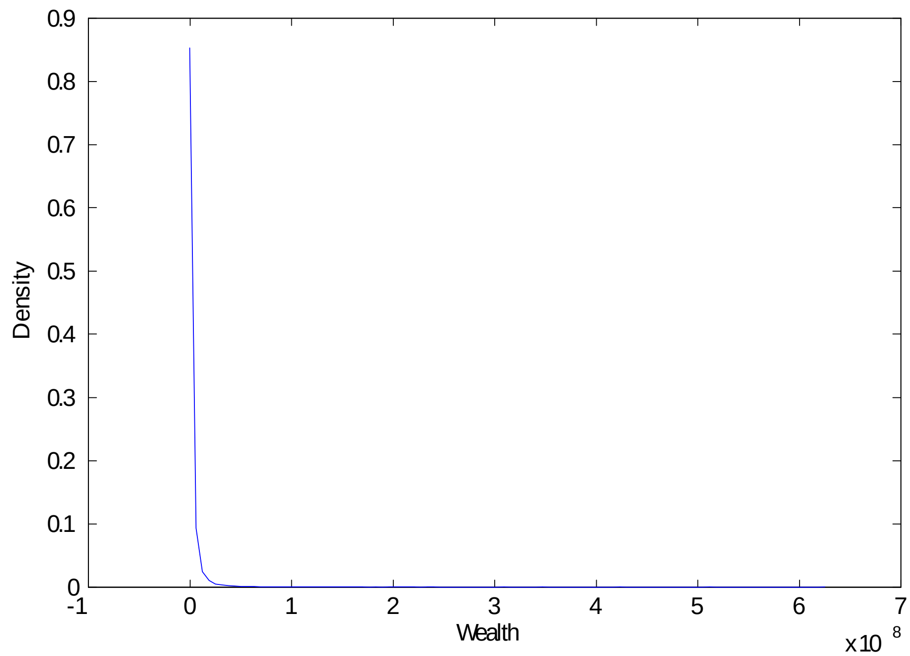
The following table shows the Gini coefficient and quintiles of the wealth distribution in U.S. and in our model economy.[1]

Economy	*Gini*	*First*	*Second*	*Third*	*Fourth*	*Fifth*
United States	0.78	−0.39	1.74	5.72	13.43	79.49
Model	0.63	1.78	5.13	9.42	17.47	66.2

To highlight the skewness to the right and heavy top tail, we further disaggregate the top groups, and compare the percentiles of the wealth distribution for U.S. and the benchmark model economy.

Economy	$90th - 95th$	$95th - 99th$	$99th - 100th$
United States	12.62	23.95	29.55
Model	13.17	20.48	16.64

[1] The data of the U.S. economy in the following two tables are from Castaneda, Diaz-Gimenez and Rios-Rull (2003) who calculate these tables from 1992 Survey of Consumer Finances (SCF).



And the Pareto exponent is

$$\mu = 1.6131$$

The simulated density funtion of wealth distribution is

The log-log plot of the density can display the Pareto upper tail

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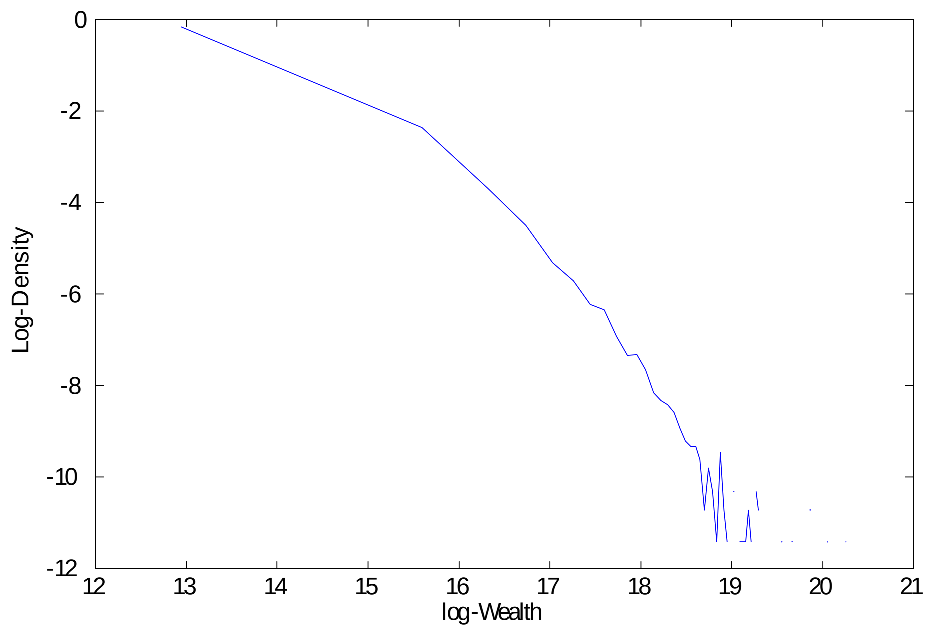
0
-2
-4
-6
-8
-10
-12
Log-Density
12
13
14
15
16
17
18
19
20
21
log-Wealth

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5. Appendix

5.1. Proof of Proposition 1.

Proof: Hamilton-Jacobi-Bellman

$$\begin{aligned}
\theta J(W,t) &= \max_{C,\omega}\{\frac{C(t)^{1-\gamma}}{1-\gamma} \\
&\quad + J_W(W,t)[rW(t) + (\alpha - r)\omega(t)W(t) - C(t) + Y(t)] \\
&\quad + \frac{1}{2}J_{WW}(W,t)\sigma^2\omega(t)^2W(t)^2 \\
&\quad + J_t(W,t)\}
\end{aligned}$$

We have the F.O.C.

$$C(t)^{-\gamma} = J_W(W,t)$$

$$J_W(W,t)(\alpha - r) = -J_{WW}(W,t)\sigma^2\omega(t)W(t)$$

Guess

$$J(W,t) = \frac{a(t)}{1-\gamma}(W(t) + b(t))^{1-\gamma}$$

where

$$b(t) = \int_t^T Y(s)e^{-r(s-t)}ds$$

Thus

$$J_W(W,t) = a(t)(W(t) + b(t))^{-\gamma}$$

$$J_{WW}(W,t) = -\gamma a(t)(W(t) + b(t))^{-\gamma-1}$$

$$J_t(W,t) = \frac{\dot{a}(t)}{1-\gamma}(W(t)+b(t))^{1-\gamma} + a(t)(W(t)+b(t))^{-\gamma}[-Y(t)+rb(t)]$$

We have

$$C(t) = a(t)^{-\frac{1}{\gamma}}[W(t)+b(t)]$$

$$\omega(t)W(t) = \frac{\alpha - r}{\gamma\sigma^2}[W(t)+b(t)]$$

Plugging these expressions into the HJB, we have

$$\begin{aligned}
&\theta\frac{a(t)}{1-\gamma}(W(t)+b(t))^{1-\gamma} \\
= \quad & a(t)^{-\frac{1-\gamma}{\gamma}}\frac{1}{1-\gamma}(W(t)+b(t))^{1-\gamma} \\
&+a(t)(W(t)+b(t))^{-\gamma}[rW(t)+\frac{1}{2}\frac{(\alpha-r)^2}{\gamma\sigma^2}(W(t)+b(t)) - a(t)^{-\frac{1}{\gamma}}(W(t)+b(t)) + Y(t)] \\
&+\frac{\dot{a}(t)}{1-\gamma}(W(t)+b(t))^{1-\gamma} + a(t)(W(t)+b(t))^{-\gamma}[-Y(t)+rb(t)]
\end{aligned}$$

Thus

$$a(t)^{\frac{1}{\gamma}-1}\dot{a}(t) + a(t)^{\frac{1}{\gamma}}[(1-\gamma)(r+\frac{1}{2}\frac{(\alpha-r)^2}{\gamma\sigma^2}) - \theta] + \gamma = 0$$

Using the boundary condition

$$a(T) = \chi(1-\zeta)^{1-\gamma}$$

we can have

$$a(t) = \left(\begin{array}{c} (\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} \\ +\left((\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} + \frac{1}{\frac{1-\gamma}{\gamma}(r+\frac{1}{2}\frac{(\alpha-r)^2}{\gamma\sigma^2})-\frac{\theta}{\gamma}}\right)\left(\exp([\frac{1-\gamma}{\gamma}(r+\frac{1}{2}\frac{(\alpha-r)^2}{\gamma\sigma^2}) - \frac{\theta}{\gamma}](T-t)) - 1\right) \end{array}\right)^{\gamma}$$

We have

$$\begin{aligned}
dW(t) &= [rW(t) + (\alpha-r)\omega(t)W(t) - C(t) + Y(t)]dt + \sigma\omega(t)W(t)dB(t) \\
&= [rW(t) + \frac{(\alpha-r)^2}{\gamma\sigma^2}(W(t)+b(t)) - a(t)^{-\frac{1}{\gamma}}(W(t)+b(t)) + Y(t)]dt \\
&\quad +\frac{\alpha-r}{\gamma\sigma}(W(t)+b(t))dB(t)
\end{aligned}$$

We know that

$$db(t) = [-Y(t) + rb(t)]dt$$

Thus

$$\begin{aligned} d(W(t)+b(t)) &= \left(r + \frac{(\alpha - r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right)(W(t)+b(t))dt \\ &\quad + \frac{\alpha - r}{\gamma\sigma}(W(t)+b(t))dB(t) \end{aligned}$$