Wealth Distribution in Finite Life with Investment Risk

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Question

- Why does the wealth distribution in U.S. displays the following characteristics?
    - A Gini coefficient as high as 0.78
    - Skewness to the right
    - Pareto tail
- I propose a parsimonious model including bequest motives and investment risk to match these three features in the data.

Three Features

- Gini and Quintiles (Castaneda, Diaz-Gimenez and Rios-Rull (2003))

Economy	*Gini*	*First*	*Second*	*Third*	*Fourth*	*Fifth*
United States	0.78	−0.39	1.74	5.72	13.43	79.49

Top tail

Economy	$90th - 95th$	$95th - 99th$	$99th - 100th$
United States	12.62	23.95	29.55

- Pareto tail. Using the richest sample of the U.S., the Forbes 400, during 1988-2003 Klass et al. (2006) find that the top end of the wealth distribution obeys a Pareto law with an average exponent of 1.49.

A continuous time OLG model

- There is a continuum of agents in the economy.
- Finite life. Each agent gives birth to one child when he dies.
  - Scenario (i) No uncertainty of life span. Thus the age cohort has equal size.
  - Scenario (ii) Age-dependent death rate. Thus the economy has realistic age cohort distribution.
- Agents may have bequest motives.
- Investment risk within lifetime. This work is different from Behabib and Bisin (2008).

Agent's problem

- $$\max_{c(t),\phi(t)} \{E_t \int_t^T \frac{c(s)^{1-\gamma}}{1-\gamma} e^{-\theta(s-t)} ds + \chi \frac{[(1-\zeta)w(T)]^{1-\gamma}}{1-\gamma} e^{-\theta(T-t)}\}$$

  $$\begin{aligned} s.t. \quad dw(t) &= [(1-\tau)rw(t) + ((1-\tau)\rho - (1-\tau)r)\phi(t)w(t) \\ &\quad -c(t) + \omega + \Gamma]dt \\ &\quad +(1-\tau)\sigma\phi(t)w(t)dz(t) \end{aligned}$$

  where ω is the wage rate, Γ is the government lump-sum transfer. τ is capital income tax rate. ζ is estate tax rate.

- The agent's human wealth

  $$h(t) = \int_t^T (\omega + \Gamma) e^{-(1-\tau)r(s-t)} ds$$

Policy functions

- The agent's policy functions are

$$c(t) = a(t)^{-\frac{1}{\gamma}}[w(t) + h(t)]$$

$$\phi(t)w(t) = \frac{(1-\tau)\rho - (1-\tau)r}{\gamma\sigma^2(1-\tau)^2}(w(t) + h(t))$$

where

$$a(t) = \left(\frac{(\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} + \left((\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} + \frac{1}{\frac{1-\gamma}{\gamma}[(1-\tau)r + \frac{1}{2}\frac{(\rho-r)^2}{\gamma\sigma^2}] - \frac{\theta}{\gamma}} \right)}{\left(\exp(\{\frac{1-\gamma}{\gamma}[(1-\tau)r + \frac{1}{2}\frac{(\rho-r)^2}{\gamma\sigma^2}] - \frac{\theta}{\gamma}\}(T-t)) - 1 \right)} \right)^{\gamma}$$

And

$$d(w(t) + h(t)) = \left((1-\tau)r + \frac{(\rho-r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}} \right)(w(t) + h(t))d$$

$$+\frac{\rho - r}{\gamma\sigma}(w(t) + h(t))dz(t)$$

Wealth accumulation within lifetime

- Let $x(t)$ be the total wealth, i.e. the sum of physical wealth and human wealth.

$$x(t) = w(t) + h(t)$$

From proposition 1, we know

$$dx(t) = \left((1-\tau)r + \frac{(\rho - r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right) x(t)dt + \frac{\rho - r}{\gamma\sigma} x(t)dz(t)$$

- The end-of-life wealth is

$$\begin{aligned} w(T) &= x(T) \\ &= (\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} a(0)^{-\frac{1}{\gamma}} \exp[(\frac{(1-\tau)r - \theta}{\gamma} + \frac{(\rho - r)^2}{2\gamma\sigma^2})T \\ &\quad + \frac{\rho - r}{\gamma\sigma} z(T)] x(0) \end{aligned}$$

Intergenerational wealth connection

- Let $T, 2T, 3T, \cdots, nT, \cdots$ be the born time of generation 1, 2, 3, $\cdots$, n, $\cdots$. Let

$$x_1 = x(T), x_2 = x(2T), x_3 = x(3T), \cdots, x_n = x(nT), \cdots$$

- Bequest Movement

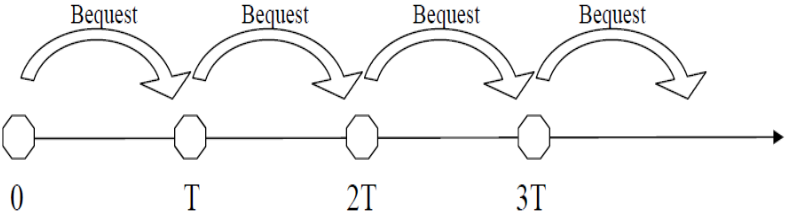


Illustration of Bequest Movement in a Linage

Equation discribing wealth connection

- Agent's starting wealth includes received bequest and human wealth

$$
\begin{aligned}
& x_{n+1} \\
= \; & (1-\zeta)w((n+1)T) + h(0) \\
= \; & \left(\frac{\chi(1-\zeta)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[(\frac{(1-\tau)r-\theta}{\gamma} + \frac{(\rho-r)^2}{2\gamma\sigma^2})T + \frac{\rho-r}{\gamma\sigma} z(T)]x_n \\
& + h(0)
\end{aligned}
$$

- Let

$$\rho_{n+1} = \left(\frac{\chi(1-\zeta)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[(\frac{(1-\tau)r-\theta}{\gamma} + \frac{(\rho-r)^2}{2\gamma\sigma^2})T + \frac{\rho-r}{\gamma\sigma} z(T)]$$

  Note that ρ_{n+1} is lognormally disributed.

Thus

$$x_{n+1} = \rho_{n+1} x_n + h(0)$$

Distribution of starting wealth

- By Sornette (2006) and Goldie (1991), the starting wealth displays an asymptotic Pareto upper tail, i.e.

$$P(x(0) > x) \sim x^{-\mu}$$

where

$$\mu = \gamma \left(\frac{\frac{1}{T} \log\left(\frac{a(0)}{\chi(1-\zeta)}\right) + \theta - (1-\tau)r}{\frac{(\rho-r)^2}{2\sigma^2}} - 1 \right)$$

Wealth distribution conditional on age

- The Pareto tail of the starting wealth distribution implies that wealth distribution conditional on any age also displays Pareto tail with the same exponent.
- Hump shape of wealth accumulation.
  If
  $$0 < \frac{\gamma - 1}{\gamma}\left((1-\tau)r + \frac{(\rho - r)^2}{2\gamma\sigma^2}\right) + \frac{\theta}{\gamma} < \frac{1}{(\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}}}$$
  then the mean wealth of the age cohort has a hump shape.

Wealth distribution of the whole economy

- The wealth distribution of the whole economy displays a Pareto tail of the same exponent as that of the starting wealth distribution.

Age-dependent death rate

- Let $\pi(t)$, $t \in [0, T]$ be the death rate of agent. Define

$$G(t) = \int_t^T \pi(s)ds$$

and

$$\pi(v,t) = \frac{\pi(v)}{G(t)}$$

- Agent's problem

$$\max_{c,P,\phi}\{E_t \int_t^T \pi(v,t)[\int_t^v \frac{c(s)^{1-\gamma}}{1-\gamma}e^{-\theta(s-t)}ds$$
$$+\chi\frac{[(1-\zeta)Z(v)]^{1-\gamma}}{1-\gamma}e^{-\theta(v-t)}]dv\}$$

$$\begin{aligned} s.t. \quad dw(t) &= [(1-\tau)rw(t) + ((1-\tau)\alpha - (1-\tau)r)\phi(t)w(t) \\ &\quad -c(t) - P(t) + \omega + \Gamma]dt \\ &\quad +(1-\tau)\sigma\phi(t)w(t)dz(t) \end{aligned}$$

Intergenerational connection

- Now let t_1, t_2, $t_3, \cdots$, $t_n, \cdots$ be the born time of generation 1, 2, 3,$\cdots$, $n, \cdots$. Let

$$x_1 = x(t_1), x_2 = x(t_2), x_3 = x(t_3), \cdots, x_n = x(t_n), \cdots$$

- We have

$$x_{n+1} = \rho_{n+1} x_n + h(0)$$

where

$$\rho_{n+1} = \frac{(\chi(1-\zeta))^{\frac{1}{\gamma}}}{a(0)^{\frac{1}{\gamma}}} \exp\{[\frac{(1-\tau)r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}](t_{n+1}-t_n) + \frac{\alpha-r}{\gamma\sigma}(z(t_{n+1})-z(t_n))\}$$

- Pareto tail

Calibrated Economy

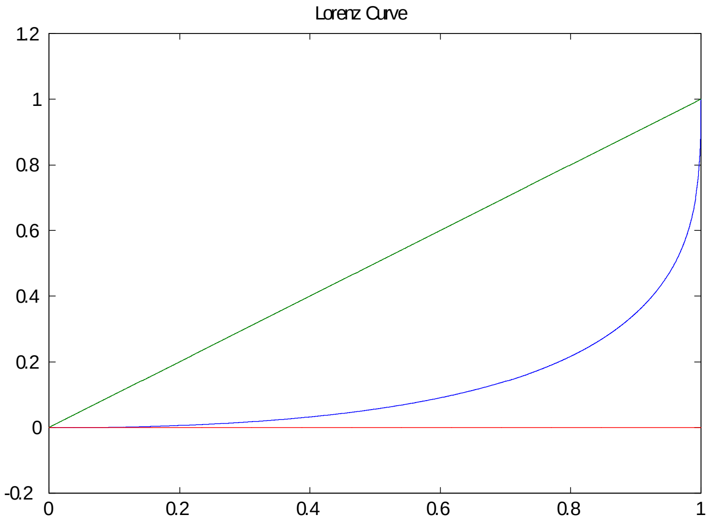
- Parameters. $\theta = 0.04$, $r = 0.01$, $\gamma = 2.5$, $\alpha = 0.08$, $\sigma = 0.2$, $\chi = 15$, $\zeta = 0.19$, $\tau = 0.25$, $t \in [20, 91]$.
- Gini and Lorenz curve.
  - Gini and Quintiles

Economy	*Gini*	*First*	*Second*	*Third*	*Fourth*	*Fifth*
United States	0.78	−0.39	1.74	5.72	13.43	79.49
Model	0.76	0.6	2.6	5.81	12.67	78.32

  - Top tail

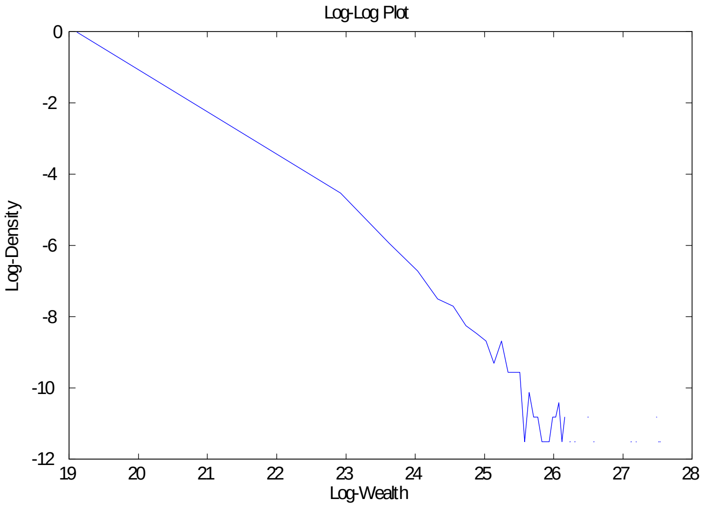
Economy	$90th - 95th$	$95th - 99th$	$99th - 100th$
United States	12.62	23.95	29.55
Model	11.9	21.34	31.91

Lorenz Curve

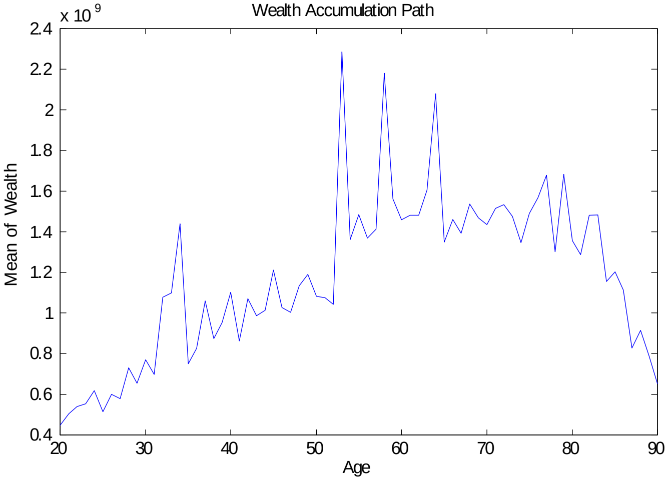


Pareto Tail

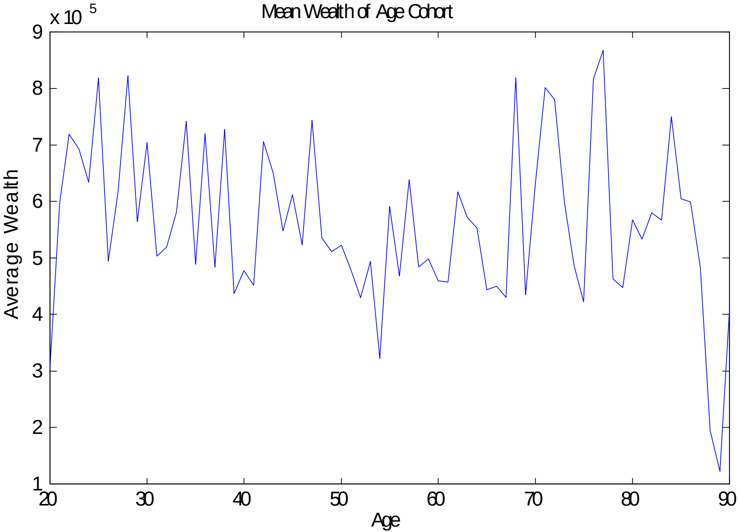
- Pareto exponent. $\mu = 1.6545$.
- Log-Log plot



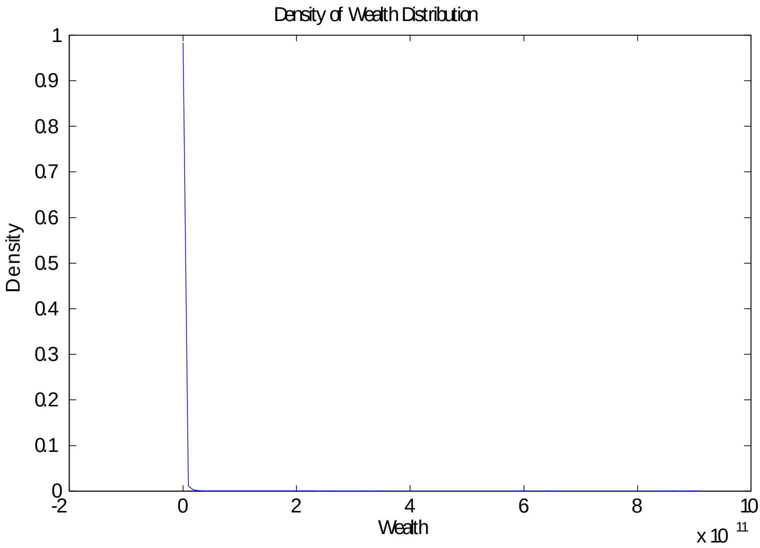
Wealth Accumulation Path



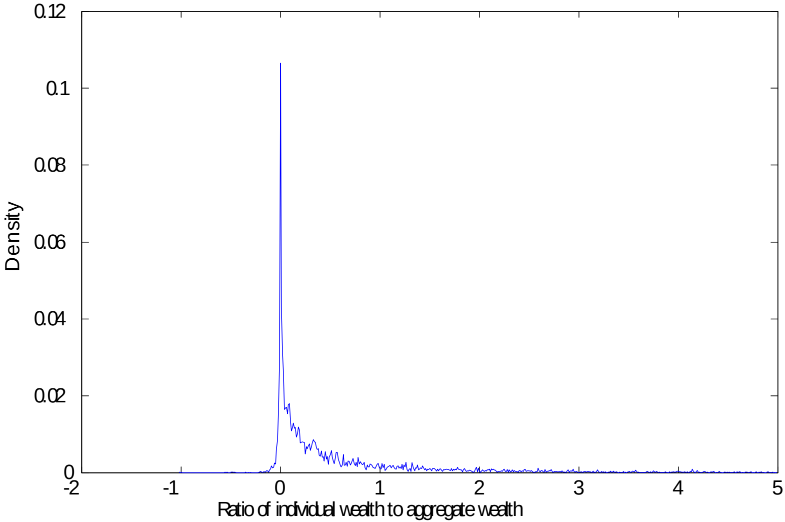
Average Wealth of Age Cohort



Density of Wealth Distribution (model)



Density of Wealth Distribution (Data)



Plan to Do

- Tax effect
- Disentangle inequality
- Wealth dispersion and consumption dispersion with aging