COMMUNICATIONS

The Golden Rule of Accumulation: A Fable for Growthmen

Once upon a time the Kingdom of Solovia was gripped by a great debate. "This is a growing economy but it can grow faster," many argued. "Sustainable growth is best," came the reply, "and that can come only from natural forces."

A few called the debate growthmanship. But most thought it would be healthy if it led to a better understanding of Solovian growth. So the King appointed a task force to learn the facts of Solovian economic life.

The committee reported that the labor force and population in Solovia grew exponentially at the rate γ. The number of working Solovians, N_t, at time t was therefore given by

$$N_t = N_0 e^{\gamma t}, \qquad \gamma > 0. \tag{1}$$

The report expressed confidence that Solovia's supply of natural resources would remain adequate. It portrayed a competitive economy making full and efficient use of its only scarce factors, labor and capital, in the production of a single, all-satisfying commodity. Returns to scale were observed to be constant, and capital and labor were found to be so substitutable that fears of technological unemployment were dismissed.

The committee described the steady progress in Solovia's ways of production. It estimated that the efficiency of Solovian capital was increasing at the rate λ and that Solovian labor was improving at the rate μ. A continuation of these rates of technical advance was anticipated. Therefore production, P_t, at time t, was the following function of available capital, K_t, and the current labor force:

$$P_t = F\left(e^{\lambda t} K_t, e^{\mu t} N_t\right), \qquad \lambda \geq 0, \mu \geq 0. \tag{2}$$

The report acknowledged further investigation of the production function might prove to be desirable.

Then the task force approached the growth issue. It doubted that technological advance could be accelerated and it took no positive stand on population increase. If γ, λ and μ were fixed parameters, then hope had to rest entirely on investment. While maintenance of the existing ratio of capital to labor would permit output per worker and per head to grow by virtue of technical progress, the report voiced the hope that higher incomes and perhaps a greater growth rate would be sought through a continuous increase in capital per worker, or what the task force called capital-deepening. It concluded by declaring the proper pace of capital-deepening to be a momentous question for Solovian political economy.

The King commended the task force for its informative and stimulating report. He invited all his subjects to join in search of an optimal investment policy. Solovian theorists considered dozens of fiscal devices for their efficiency, equity and effectiveness. Mathematicians, leading the quest for a growth strategy, grappled with extremals, functionals and Hamiltonians. Yet nothing practicable emerged.

Then a policy-maker was heard to say, "Forget grand optimality. Solovians are a simple people. We need a simple policy. Let us require that the fraction of output accumulated be fixed for all time, that is:

$$\frac{dK_t}{dt} = sP_t, \text{ for all } t, 0 \leq s \leq 1. \tag{3}$$

If we make investment a constant proportion of output, our search for the idea' investment policy reduces to finding the best value of s, the fixed investment ratio."

"It's fair," Solovians all said. The King agreed. So he established a prize for the discovery of the optimum investment ratio. The prize was to be a year abroad to learn how advanced countries had solved the growth problem.

Soon a brilliant peasant, Oiko Nomos, claimed the prize. Solovians laid down their tools, picked up pencils and pads, and converged on their capital to hear the proposed solution.

Oiko spoke. "I begin with a definition. By a *golden age* I shall mean a dynamic equilibrium in which output and capital grow exponentially at the same rate so that the capital-output ratio is stationary over time. This is precisely the pattern of growth which might emerge asymptotically from the regime contemplated for Solovia where population growth and technical progress are expected to be exponential and the investment ratio is to be fixed for all time.

"Now I am obliged to make some assumptions which I hope, later researches into the exact shape of our production function will support:

"First, I assume that Solovia is capable of golden-age growth. This simply means that, corresponding to every investment ratio Solovia might adopt, there exists at least one capital-output ratio which, if established, will be exactly maintained by the dynamic equilibrium which follows from equations (1)-(3).

"Second, I assume that Solovia's golden-age growth rate is independent of its investment ratio We may call this growth rate, g, the *natural* rate of growth, in that it depends not upon our investment decisions but only upon γ, λ, μ and possibly certain parameters affecting the shape of the production function. The existence of a natural growth rate implies capital and labor are substitutable in such a way that the capital-output ratio can adjust to any value of s so as to equate the rate of capital growth,

$$\frac{sP_t}{K_t}, \text{ to the natural rate of output growth, } g.$$

"We can express the output of an economy in a golden age and having a natural growth rate by the equation:

$$P_t = P_0 e^{gt}, \qquad g > 0 \tag{4}$$

where P_0 depends upon conditions at time zero.

"We come now to a crucial notion. Consider an economy which lacks a definite beginning and which has always enjoyed golden-age growth at the natural rate. It has traveled unswervingly up a single exponential path, a path stretching back indefinitely into the past. Along this path the output rate at any specified time (though not the rate of growth) depends, in general, upon the value of the equilibrium capital-output ratio. But this ratio depends upon the investment ratio that has reigned over the golden age; we noted earlier that under conditions of natural growth the capital-output ratio is simply:

$$\frac{K_t}{P_t} = \frac{s}{g}. \tag{5}$$

Therefore, the golden-age output rate at any time—the height of the growth path—is generally a function of the prevailing value of s. We can express this fact by replacing P_0 in (4) by the function $f(s)$. Thus:

$$P_t = f(s)e^{gt}. \tag{6}$$

"It has been observed that a large value of s corresponds to a small ratio of output to capital. Provided that the elasticity of output with respect to capital is uniformly smaller than one, a seeming condition for stability, the smaller the ratio of output to capital, the larger must be the absolute magnitudes of both output and capital. Hence $f'(s) > 0$.

"I shall call a golden age which lacks a definite beginning a *boundless golden age*. Such an age may be endless although that is not essential for the definition; but it must be endless looking backward.

"And now, if these concepts are clear and my assumptions granted, I wish to introduce the following lemma."

"A lemma, a lemma," the crowd shouted. It was plain that the Solovians were excited by the prospect.

Oiko resumed. "The lemma: *Each generation in a boundless golden age of natural growth will prefer the same investment ratio, which is to say the same natural growth path.*

"In deciding which growth path is best from its standpoint, a generation will look only at the amount of consumption which each path offers it. Given the constancy of s, every golden-age path is associated with a consumption path on which consumption grows exponentially at the same rate as output. Under conditions of natural growth, consumption along all these paths grows at the identical rate, g, so that these time paths of consumption cannot cross. Therefore, with resources limited, there must exist some uniformly highest, feasible consumption path. This dominant consumption path offers more consumption at every point in its history than any other natural-growth consumption path. All generations in such a history will naturally prefer this path, whence its corresponding investment ratio, to any lower consumption path. A rigorous demonstration is straightforward.

"Take the consumption rate of the 'generation' in a boundless and natural golden age at time t. By (3) and (6), this is:

$$C_t = (1\text{-}s)f(s)e^{gt}. \tag{7}$$

To find the value of s which maximizes C_t, we take the derivative with respect to s and equate it to zero. This yields:

$$-f(s)e^{gt} + (1\text{-s})\, f'(s)e^{gt} = 0. \tag{8}$$

"It is apparent that upon dividing (8) by e^{gt} all terms involving t vanish. The solution of equation (8) is therefore independent of the 'generation' whose consumption we choose to maximize. The s which is optimal for one generation in a natural boundless golden age is optimal for all. This proves the lemma."

Cries of "What a lemma!" resounded in the capital and Oiko was heartened by the reception. Anticipation ran high when he moved to speak again.

"And now I wish to announce a new and fundamental theorem. Theorem: *Along the optimal golden-age path, under conditions of natural growth, the rate of investment is equal to the competitive rate of profits.*

"Choosing the best value of s is simple enough in principle. A high value of s will be associated with a high golden-age output path. But too high a value of s will leave too little output available for consumption. Characterizing the exact optimum is a matter of calculus.

"Rewriting (8) in the form:

$$\frac{s}{1\text{-}s} = \frac{f'(s)s}{f(s)} \tag{8'}$$

we find that the optimal ratio of investment to consumption equals what we may call the elasticity of golden-age output at time zero with respect to the investment ratio. Looking at (6), it is obvious that, for every investment ratio, this elasticity must be the same at all points (dates) along the associated golden-age path. If this were not so, the golden-age growth rate would depend upon the investment ratio, contrary to our assumption of natural growth.

"The remaining task is to express this elasticity in explicit terms of the production function, and thus in terms of relative factor shares.[1] Now the production function indicates that $f(s) = F(K_0, N_0)$. Next we use the golden-age capital-output relation in (5) to write K_0 in the form $\frac{sP_0}{g}$. Upon making this substitution in the production function (2) we obtain an equation in golden-age output at time zero as function of itself, the investment ratio and the labor force:

$$f(s) = F\left(\frac{sf(s)}{g}, N_0\right). \tag{9}$$

"Total differentiation of (9) with respect to s yields an equation in terms of $F_K(K_0,N_0)$, the marginal productivity of capital at time zero:

$$f'(s) = F_K\,\frac{f(s)}{g} + F_K\,\frac{s}{g}\,f'(s). \tag{10}$$

[1] Oiko was seen at this point to wave gratefully to Richard Nelson for help with this proof.

Upon rearranging terms and using the capital-output relation (5) we find that

$$\frac{f'(s)s}{f(s)} = \frac{a}{1\text{-}a}, \text{ where } a = \frac{F_K(K_0, N_0)K_0}{P_0}. \tag{11}$$

"Looking at (8′) and (11) we see easily that

$$s = a \tag{12}$$

In competitive Solovia the variable a measures capital's relative share in total output at time zero. Now we have observed that the elasticity of golden-age output with respect to the investment ratio is everywhere equal on any particular golden-age path; it follows by (11) that a, the profit-income ratio, must also be constant along any particular golden-age path. Therefore, by (12), on the optimum natural growth path the investment ratio and the profit ratio are constant and equal. This proves the theorem.

"We may call relation (12) the *golden rule of accumulation,* and with good reason. In a golden age governed by the golden rule, each generation invests on behalf of future generations that share of income which, subject to (3), it would have had past generations invest on behalf of it. We have shown that, among golden-age paths of natural growth, that golden age is best which practices the golden rule."

The Solovians were deeply impressed by Oiko and his theorems. But they were a practical people and soon full of queries. How, Oiko, does your theorem apply to Solovia? What must we do if we are not already on the golden-age, golden-rule path? Should we abide by the golden rule even when out of golden-rule equilibrium?

"Perhaps," Oiko replied. "We might attempt to approach the golden-rule path asymptotically. However I urge that we, in our lifetime, take whatever steps are required to place Solovia securely on the golden-rule path. Associated with that path is a unique capital-output ratio. If our present capital-output ratio is smaller, then our consumption must be slowed until our ratio is no longer deficient. If our present ratio exceeds the golden-rule ratio, then we must consume faster until our capital-output ratio is no longer excessive.

"Once our capital-output ratio has attained its golden-rule value, we must make a solemn compact henceforth to invest by the golden rule. If the investment ratio remains ever equal to the profit ratio, no generation in all the future of Solovia will ever wish we had chosen a different, successfully enforced investment ratio. The foundations are thus laid for a quasi-optimal social investment policy."

The crowd dispersed, happy for their Kingdom's future. But there were skeptics who reminded the King of Oiko's assumptions. They questioned Solovia's immunity from technological unemployment. They wondered whether their production function admitted of a natural growth rate. So the King named a team of econometricians to investigate the shape of the Solovian production function.

The King's econometricians were eventually satisfied that production in Solovia took place according to the Cobb-Douglas function:

$$P_t = A(e^{\lambda t}K_t)^{\alpha}(e^{\mu t}N_t)^{1-\alpha} \qquad 0 < \alpha < 1 \tag{2′}$$

where α, a fixed parameter, was the elasticity of output with respect to the capital stock. They preferred to write it in the form:

$$(2'') \quad P_t = Ae^{\rho t} K_t^{\alpha} N_t^{1-\alpha}, \qquad \text{where} \qquad \rho = \alpha\lambda + (1\text{-}\alpha)\mu.$$

Solovians knew then they could have any capital-output ratio they desired, with full employment. The existence of a full-employment, golden-age equilibrium for every investment ratio was assured. Differentiating logarithmically, they quickly calculated from (1) and (2″) that in a golden age, capital and output would grow exponentially at the rate $\frac{\rho + (1\text{-}\alpha)\gamma}{1\text{-}\alpha}$, independently of the investment ratio. Thus did Solovia discover her natural rate of growth. What a triumph for Oiko. His assumptions were completely vindicated.

Joyously, the Solovians hurried to compute the golden-rule path. It did not take them long to realize that α was capital's share. On the golden-rule path, s would equal α. Next, using (5), they divided α by their natural growth rate to obtain the capital-output ratio on the golden-rule path. To their great relief, the resulting ratio exceeded their actual capital-ouput ratio by only a small factor. No wonder for they had invested most of their profits and consumed most of their wages anyway.

With Oiko's inspiring words still ringing in their ears, the Solovian people pressed the King for a program to attain the golden-rule path. So the King proclaimed golden-rule growth a national purpose and instituted special levies. Once the golden-rule path was reached, investment was continuously equated to profits and Solovians enjoyed, subject to (3), maximum social welfare ever after.

EDMUND PHELPS*

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The Bethlehem-Youngstown Case and the Market-Share Criterion

One of the most important unsolved problems in antitrust policy concerns the precise significance of the market share of the defendant in determining whether or not certain violations have occurred. Under Section 2 of the Sherman Act [1], it is apparently no longer true (if it ever was) that one-firm production of some (unspecified) large percentage of any more or less homogeneous product is usually sufficient to prove violation and justify dissolution.