Pareto distribution in growth models and the U.S. inequalization

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Abstract

This paper demonstrates that the income distribution converges to a Pareto distribution in a variation of the Solow growth model in which households hold heterogenous asset positions and bear idiosyncratic shocks on asset returns. The Pareto exponent of the tail distribution is derived explicitly. The exponent is determined by the balance between the savings from the labor income and the variance of the asset returns. It is shown that the excess returns of risky assets which hit the bottom around 1970 may explain the inverted-U curve of the historical US Pareto exponent. The effect of zero or negative personal savings on Pareto exponent is also investigated.

1 Introduction

It is well known that the tail distributions of income and wealth follow a Pareto distribution in which the frequency decays in power as $\Pr(x) \propto x^{-\xi-1}$. There is a consensus that the Pareto distribution is generated by multiplicative shocks in the wealth accumulation process. We formalize this idea in the framework of standard models of economic growth.

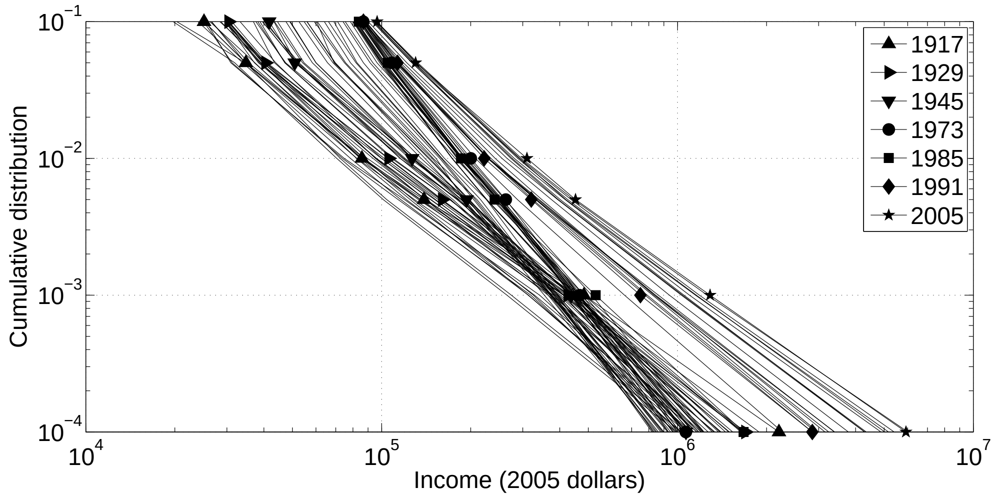

Figure 1: Income distributions above the top ten percentile

This paper is motivated by the recent study by Feenberg and Poterba [5] and Piketty and Saez [15] who investigated the time series of top income share by using tax returns data. Figure 1 plots the income distributions from 1917 to 2006 that are exhibited in the data on the top income share compiled by Piketty and Saez [15]. The distribution is shown in log-log scale and cumulated from above. The plot shows only the tail portion of the distribution (above the 90th percentile), which is our main concern in this paper.

We observe that the distributions in 1917 and 2006 are nearly parallel. This means that the income at each percentile grows at the same average rate in the periods. However, the growth rates were not constant during the 90 years. We observe that the slope is steepened during the post-war period until 1970s, and since then it is flattened. That is, the income at the 90th percentile grew mostly in the first 50 years and stagnated since then, whereas the growth of income at the 99.9th percentile mostly occurred in the last 40 years.

We note that a linear line well fits the distribution in the double-log scale, $\log \Pr(X > x) = a - \xi \log x$, where the slope ξ is called Pareto exponent. The distribution is more equal when ξ is larger. We estimate ξ by linear fits and plot its time series in Figure 2.

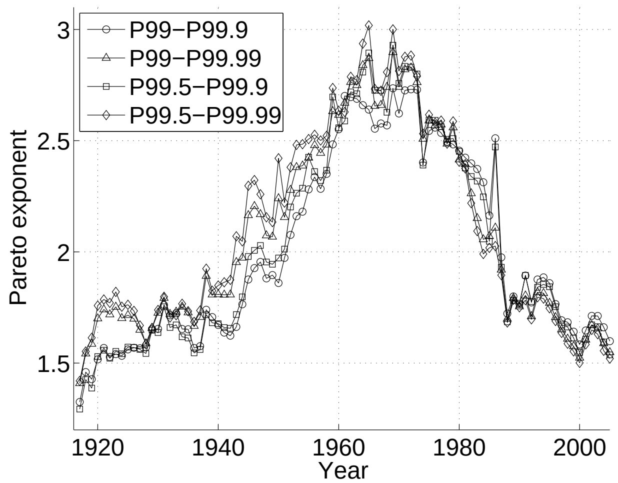

Figure 2: Pareto exponents for the U.S. income distributions

There is a relatively large estimation error for ξ since it is determined by the relatively small number of households. To account for the inaccuracy, we estimate ξ by different range of the tail as seen in Figure. Although there are some variation of estimated ξ depending on the range we pick, there is a clear trend of ξ seen in the plot: it shows the Kuznets' inverted U – or rather V – pattern. Pareto exponent has increased steadily until 1960s, and since then declined fairly monotonically. There is a clear switch in the trend around 1970.

We also note that the exponents stay in the range 1.5 to 3. The stability of ξ confirms Kalecki's observation [9] that the income distribution does not exhibit an ever-increasing log-variance. The range, 1.5 to 3, is also consistent with the estimates from different economies in different periods. For example, the income distribution tail in Japan exhibits the stable Pareto exponent around 2 as shown in Figure 3. Such a stability of ξ in various environments suggests some universality in the determinant of ξ.

This paper investigates the universality of the Pareto distribution and its exponent, and the trend break in the U.S. inequality. We develop a simple theory of income distribution based on the Solow growth model. Incorporating an idiosyncratic asset return shock,

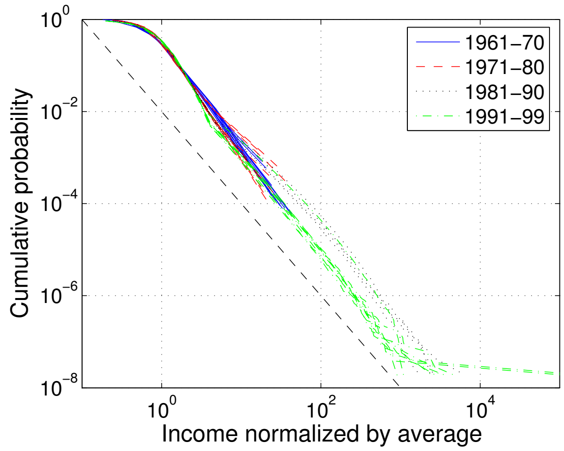

Figure 3: Income distributions in Japan (Source: Fujiwara et al. [6]. Graph is taken from Nirei and Souma [14].)

the Solow model generates a stationary Pareto distribution for the detrended household income at the balanced growth path. Pareto exponent ξ is analytically determined by fundamental parameters. The determinant of ξ is summarized by the balance between the inflow from the middle class into the tail part and the inequalization force within the tail part. The inflow is represented by the savings from wage income and the inequalization force is caputured by the capital income that is attributed to risk taking behaviors of the top income earners. When the two variables are equal, Pareto exponent is determined at $\xi = 2$.

The model has implications for the U.S. developments of inequality. Among the factors that affect the Pareto exponent, the labor income share and the capital-output ratio are fairly stable. The risk premium seems to exhibit a long run movement, as it declined until 1970s and backed up since then. We argue that our analysis makes a good link between the movements in the risk premium and the Pareto exponent. Another variable that exhibits a long run movement is the personal savings rate. Although the savings rate does not affect the Pareto exponent at the balanced growth path in our model, it matters during the transition, and it can have a dramatic impact on ξ if it becomes zero or negative.

Our analysis complements the finding of Piketty and Saez [15]. They established that the top income share was decreased dramatically by the great depression and the second world war, stayed at the low level until it started creeping up around 1980s. The top income share they focued is determined by the growth of the top income group relative to the entire economy. To contrast, the Pareto exponent represents the inequality among the top income earners. In other words, the top income share traces the intercept of the tail distribution; here we characterize its slope. Our analysis of the tail augments the understanding of the underlying mechanism for the overall inequality.

There is a long tradition of theoretical works for Pareto distributions. Gibrat first argued that an accumulation process with independent growth rates generate a lognormal distribution of wealth, which is right-skewed and has a long tail. Kalecki [9] pointed out that the variance of log income did not grow linearly in time, contrary to the Gibrat's hypothesis for the lognormal development. Empirical estimates of Pareto exponent ξ often fall in the range between 1.5 and 3, whereas the log-normal development implies an ever falling tail exponent. Subsequent studies by Champernowne [4], Ijiri and Simon [8], and Rutherford [16], among others, show that a slight modification of the multiplicative process can generate the Pareto distribution. Recently, Levy and Solomon [11] and Gabaix [7] showed that a reflexive lower bound for a multiplicative process can have a similar effect, based on the work of Kesten [10]. The present paper extends this line of research by articulating the economic meanings of the multiplicative shocks and the reflexive lower bound in the context of the standard growth model.

2 Solow model

In this section, we present a Solow growth model with an uninsurable and undiversifiable investment risk. Consider a continuum of infinitely-living consumers $i \in [0, 1]$. Each consumer is endowed with one unit of labor, an initial capital $k_{i,0}$, and a "backyard" production technology that is specified by a Cobb-Douglas production function:

$$y_{i,t} = k_{i,t}^{\alpha}(a_{i,t}l_{i,t})^{1-\alpha} \tag{1}$$

where $l_{i,t}$ is the labor employed by i and $k_{i,t}$ is the capital owned by i. The labor-augmenting productivity $a_{i,t}$ is an i.i.d. random variable across households and across periods with a common trend $\gamma > 1$:

$$a_{i,t} = \gamma^t \epsilon_{i,t} \tag{2}$$

where $\epsilon_{i,t}$ is a random variable with mean 1. Households do not have a means to insure against the productivity shock $\epsilon_{i,t}$ except for its own savings. Note that the effect of the shock is temporary. Even if i drew a bad shock in $t-1$, its average log productivity is flushed to the trend level γ^t in the next period. Thus, we can interpret that households can diversify there trend risk whereas they cannot diversity a temporary risk.

The capital accumulation follows:

$$k_{i,t+1} = (1-\delta)k_{i,t} + s(\pi_{i,t} + w_t) \tag{3}$$

where s is a constant saving rate and $\pi_{i,t} + w_{i,t}$ is i's income. Here we assumed that each household supplies one unit of labor inelastically. $\pi_{i,t}$ denotes the profit from the production:

$$\pi_{i,t} = \max_{l_{i,t}, y_{i,t}} y_{i,t} - w_t l_{i,t} \tag{4}$$

subject to the production function (1).

First order condition of the profit maximization yields a labor demand function:

$$l_{i,t} = \left(\frac{1-\alpha}{w_t}\right)^{1/\alpha} a_{i,t}^{(1-\alpha)/\alpha} k_{i,t} \tag{5}$$

Plugging into the production function, we obtain:

$$y_{i,t} = (1-\alpha)^{(1-\alpha)/\alpha} a_{i,t}^{(1-\alpha)/\alpha} w_t^{(\alpha-1)/\alpha} k_{i,t} \tag{6}$$

and $\pi_{i,t} = \alpha y_{i,t}$.

The equilibrium condition for the labor market is $1 = \int_0^1 l_{i,t} di$. Then we obtain the equilibrium wage as:

$$w_t = (1-\alpha)E(a_{i,t}^{(1-\alpha)/\alpha})^\alpha K_t^\alpha \tag{7}$$

where $K_t \equiv \int_0^1 k_{i,t} di$.

Substituting $\pi_{i,t}$ and w_t in the capital accumulation equation (3) and aggregating across households, we obtain the following equation of motion for the aggregate capital:

$$K_{t+1} = (1-\delta)K_t + sE(a_{i,t}^{(1-\alpha)/\alpha})^{\alpha} K_t^{\alpha} \tag{8}$$

Define a detrended aggregate capital as:

$$X_t \equiv \frac{K_t}{\gamma^t} \tag{9}$$

Then the equation of motion is written as:

$$\gamma X_{t+1} = (1-\delta)X_t + \tilde{s}X_t^{\alpha} \tag{10}$$

where,

$$\tilde{s} \equiv sE(\epsilon_{i,t}^{(1-\alpha)/\alpha})^{\alpha} \tag{11}$$

Equation (10) shows the deterministic dynamics for X_t. The steady state value of X_t is:

$$\bar{X} = \left(\frac{\tilde{s}}{\gamma - 1 + \delta}\right)^{1/(1-\alpha)} \tag{12}$$

The steady state is stable and unique in the domain $X > 0$. Thus, the detrended aggregate capital converges to $\bar{X}$ deterministically. The characteristics of the aggregate capital is similar to those of the Solow model. It is easily verified that the golden rule savings rate is the same as in the Solow model.

Next we consider the dynamics of individual capital. Define a detrended individual capital:

$$x_{i,t} \equiv \frac{k_{i,t}}{\gamma^t} \tag{13}$$

Substituting in (3), we obtain:

$$\gamma x_{i,t+1} = \left(1 - \delta + \tilde{s}\alpha X_t^{\alpha-1}\epsilon_t^{(1-\alpha)/\alpha}/\mathrm{E}(\epsilon_t^{(1-\alpha)/\alpha})\right) x_{i,t} + \tilde{s}(1-\alpha)X_t^{\alpha} \tag{14}$$

The system of equations (10,14) defines the dynamics of (x_t, X_t). The dynamics of X_t is deterministic, and converges to the steady state value $\bar{X}$. At the steady state, the dynamics of an individual capital $x_{i,t}$ (14) becomes:

$$x_{i,t+1} = g_{i,t}x_{i,t} + z \tag{15}$$

where

$$g_{i,t} \equiv \left(\frac{1-\delta}{\gamma} + \frac{\alpha(\gamma-1+\delta)}{\gamma} \frac{\epsilon_{i,t}^{(1-\alpha)/\alpha}}{\mathrm{E}(\epsilon_{i,t}^{(1-\alpha)/\alpha})} \right) \tag{16}$$

$$z \equiv \frac{\tilde{s}(1-\alpha)}{\gamma} \left(\frac{s}{\gamma-1+\delta} \right)^{\alpha/(1-\alpha)} \tag{17}$$

Equation (15) is called a Kesten process, which is a stochastic process with a multiplicative shock and an additive constant. At the steady state,

$$\mathrm{E}(g_{i,t}) = 1 - z/\bar{x} \tag{18}$$

must hold, where the steady state $\bar{x}$ is equal to the aggregate steady state $\bar{X}$. $\mathrm{E}(g_{i,t}) = \alpha + (1-\alpha)(1-\delta)/\gamma < 1$ holds by the definition of g (16), and hence the Kesten process is stationary.

The following proposition is obtained by a direct application of the theorem shown by Kesten [10] (see also [11] and [7]):

Proposition 1 *$x_{i,t}$ has a stationary distribution whose tail follows a power decay:*

$$\Pr(x_{i,t} > x) \propto x^{-\xi} \tag{19}$$

where the exponent of the power-law ξ is determined by the condition:

$$\mathrm{E}(g_{i,t}^{\xi}) = 1 \tag{20}$$

The exponent ξ is called a Pareto exponent, acknowledging Pareto's original discovery of the regularities in the tails of income distributions. The stationary distribution of $x_{i,t}$ has a finite mean if $\xi > 1$ and a finite variance if $\xi > 2$. Since $\mathrm{E}(g_{i,t}) < 1$, it immediately follows that the stationary distribution has a finite mean in our case. If ξ is found in the range between 1 and 2, the detrended capital distribution has a finite mean but an infinite variance. In an economy with finite households, the infinite variance implies that the population variance grows unboundedly as the population size grows.

Pareto distribution implies the self-similarity of income distribution. Suppose that you belong to a "millionaire club" where all the members earn more than a million. In the club, you find that $10^{-\xi}$ of the club members earn 10 times more than you. If $\xi = 2$, this is one percent of the all members. Suppose that, after some hard work, you now belong to a ten-million earners' club. But then, you find again that one percent of the club members earn 10 times more than you do. If your preference for wealth is ordered by the relative position of your wealth among your social peers, then you will never be satisfied by climbing up the social ladder of millionaire clubs.

Since ξ determines the existence of moments of the distribution, it implies not only a quantitative but also a qualitative characteristic of the economic equality. If $\xi = 1$, the mean income is infinite. $\xi = 1$ also implies that the income of the maximum earner is of order N^0 where N denotes the population. This means that the maximum earner obtains a fraction of the total income even when N tends to infinity. This almost resembles an aristocratic society where the fraction of total wealth is owned by a single person. It is known that the firm size and city size distribution exhibits $\xi = 1$, which is also called as Zipf's law.

If $\xi = 2$, the mean income is finite but the variance is infinite. Empirical income distributions often exhibit the exponent around 2. One economic interpretation of this exponent is that the share of risk bearing is distributed very unequally. In our economy, the households do not diversify the investment risk. Thus the variance of their income increases proportionally to the square of their wealth $x_{i,t}^2$. This variance is distributed so unequally that the most wealthiest bears a fraction of the sum of the variances of the idiosyncratic risks. Thus, if the observed fat tail of income is indeed generated by the idiosyncratic asset return risks as in our model, then the concentration of the wealth among the richest can be interpreted as a reward for their extraordinary risk bearing.

Empirical studies have found that ξ is usually found in the range between 1.5 and 3. This implies that the economy goes back and forth between the two regimes, one with finite variance of income ($\xi > 2$) and one with infinite variance ($\xi \leq 2$). We can obtain an intuitive characterization for the threshold point $\xi = 2$. At the point, $\mathrm{E}(g_{i,t}^2) = 1$ must hold by (20). Using $\mathrm{E}(g_{i,t}) = 1 - z/\bar{x}$, this leads to the condition $\mathrm{Var}(g_{i,t})/2 = z/\bar{x} - (z/\bar{x})^2/2$. We summarize the finding as follows.

Proposition 2 *The Pareto exponent ξ is strictly greater than 1. Moreover, ξ is less than or more than 2 depending on whether the variance of the returns to assets* $\mathrm{Var}(g_{i,t})$ *is* *more than or less than* $2z/\bar{x} - (z/\bar{x})^2$.

Provided that the variance of the wealth growth is significantly smaller than 1, we can neglect the second order term $(z/\bar{x})^2$ and obtain $z = \bar{x}\mathrm{Var}(g_{i,t})/2$ as the condition for $\xi = 2$. The right hand side expresses the growth of capital due to the diffusion effect. As we argue later, we may interpret this term as the capital income due to the risk taking behavior. The left hand side z represents the savings from the labor income. Then, the Pareto exponent is determined at 2 when the contribution of labor to capital accumulation balances with the contribution of risk taking. In other words, the stationary distribution of income exhibits a finite or infinite variance depending on whether the wage contribution to capital accumulation exceeds or falls short of the contribution from risk takings.

The key variable $z/\bar{x}$ is equivalently expressed as $z/\bar{x} = (s/\gamma)\bar{W/K} = (s(1-\alpha)/\gamma)\bar{Y/K}$. If we apply the commonly accepted values for the parameters in the U.S. such as $s = 0.1$, $\alpha = 1/3$, $\gamma = 1.04$, and $\bar{Y/K} = 1/2$, then we obtain $z/\bar{x} = 1/31$. Then, according to our formula, $\xi = 2$ occurs when the standard deviation of $g_{i,t}$ is about 0.25.

The formula for a general ξ is obtained when g follows a lognormal distribution. The condition (20) and the stationarity condition (16) become:

$$\begin{aligned} 0 &= \mathrm{E}(\log g) + \xi \mathrm{Var}(\log g)/2 && (21) \\ \log(1 - z/\bar{x}) &= \mathrm{E}(\log g) + \mathrm{Var}(\log g)/2 && (22) \end{aligned}$$

Then we obtain

$$\xi = 1 - (2/\mathrm{Var}(\log g)) \log(1 - z/\bar{x}) \tag{23}$$

as in Gabaix [7]. We observe that the Pareto exponent ξ declines to one as the growth rate variance ($\mathrm{Var}(\log g)$) increases to infinity. Using the same conditions, we obtain an alternative expression:

$$\xi = -\mathrm{E}(\log g)/(\mathrm{Var}(\log g)/2) \tag{24}$$

This indicates that ξ is determined by the relative importance of the trend and the diffusion of the capital growth rates that both contribute to the overall growth rate.

The crux of our mechanism for Pareto distribution is that the savings from wage income serves as a reflexive lower bound of the multiplicative wealth accumulation. The multiplicative process is the most natural mechanism for the right-skewed, heavy tailed distribution of income as the extensive literature on the Pareto distribution indicates. Without some modification, however, the multiplicative process leads to a lognormal development and does not generate the Pareto distribution nor the stable variance of log income. Incorporating the wage income in the accumulation process just does this modification. Moreover, we find that the Pareto exponent is determined by the balance between the contributions of this additive term (savings) and the diffusion term (capital income).

Finally, the savings from the wage income determines the mobility between the tail wealth group and the rest. Thus, we can interpret our result as the mobility between the top and the middle sections of income determines the Pareto exponent. This point makes a contrast to the other measurement of inequality such as the top income share.

3 Onset of the U.S. inequalization

Figure 2 showed that the Pareto exponent has experienced a large change in drift around 1970s in the U.S. It has been recognized that the income distribution has become inegalitarian since 1980s ([15], [18]), and it has come to much public notice. The flattened tail constitutes an important part of this inequalization process. In this section, we explore the implications of our model on this U.S. experience.

Our model identifies several fundamental parameters as the determinant of Pareto exponent. Let us focus on the key variable, $(z/\bar{x})/(\mathrm{Var}(g)/2)$, that measures the ratio between the contributions of the savings from labor income and of the diffusion effect. The numerator can be decomposed into the personal savings rate, the labor share of income, and the output-capital ratio. The latter two factors are fairly stable, while the personal savings rate shows a steady decline in these years. Figure 4 plots the historical personal savings rate in the NIPA statistics. The denominator, the variance of the asset growth rate for individuals, is hard to measure. Figure 4 plots the time series of the annual excess returns of the S&P500 index relative to the treasury bills (smoothened by 7-year moving average). The plotted excess returns measure our $\mathrm{Var}(g)/2$ under the

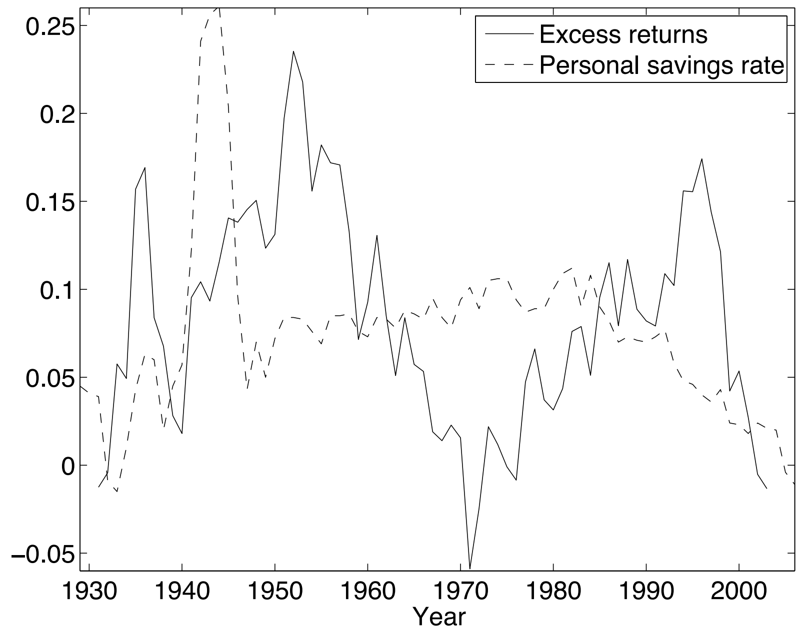

Figure 4: Excess returns measured by S&P500 index returns less T-Bills return (left axis) and the personal savings rate (right axis). The excess returns are smoothed by 7-year moving average.

assumptions that the individual wealth experiences the same volatility as the S&P500 index, and that the mean of the logarithmic instantaneous returns coincide between the S&P500 index and the Treasury Bills. Under the assumptions, the logarithm of the annual return of the S&P500 is equal to the returns on the treasury bills plus a half of the logarithmic variance of the S&P500 returns.

We note that the excess returns experienced a marked trough around 1970. Figure 5 shows the model prediction when the excess return is used for $\mathrm{Var}(g)/2$ and when the NIPA personal savings rate is used to compute z in our formula for ξ (23). The time series of the labor share is computed from the NIPA statistics and the capital-output ratio is fixed at the historically stable value, 2 (Maddison [12]). The predicted Pareto exponent is mostly too small to match the estimated exponent, possibly due to the underestimation

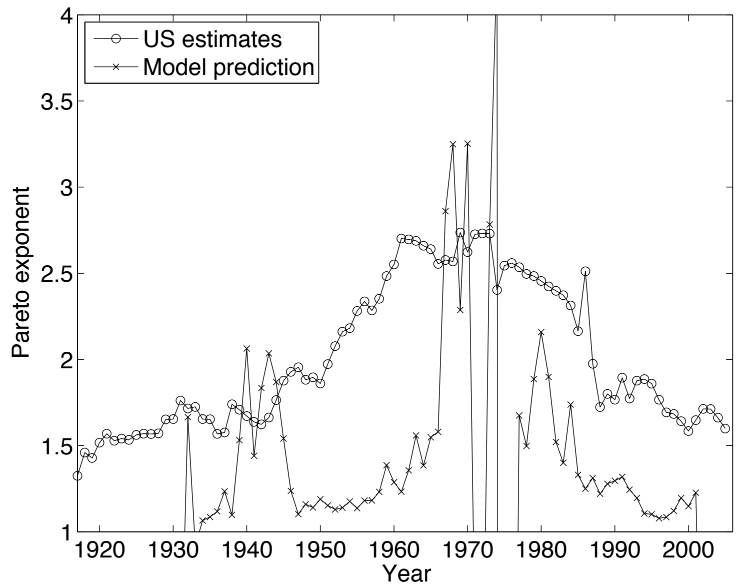

Figure 5: US Pareto exponent (estimated in the range of the 99 percentile to the 99.9 percentile income) and the model prediction when the excess returns of S&P500 index is used as the proxy for Var(g)/2.

of the contribution from the savings and due to the overestimation of the wealth growth variance. Also, the predicted exponent becomes extremely large or negative around 1970, due to the near-zero or negative values of the excess returns of S&P500 index during the periods. However, the movement of the predicted series is remarkably parallel to the actual series.

The inverted-U pattern of the predicted series is largely caused by the movement of the excess returns. Since the formula (23) applies only for the stationary distribution of income, we need to examine whether the convergence of the income distribution upon the change of the wealth growth variance is fast enough so that we can possibly explain the decade-by-decade movement of Pareto exponents by the movement of growth variance. We check the speed of convergence by simulating the model. We first compute the economy with one million households with parameter values $\gamma = 1.03$, $\delta = 0.1$, $\alpha = 1/3$,

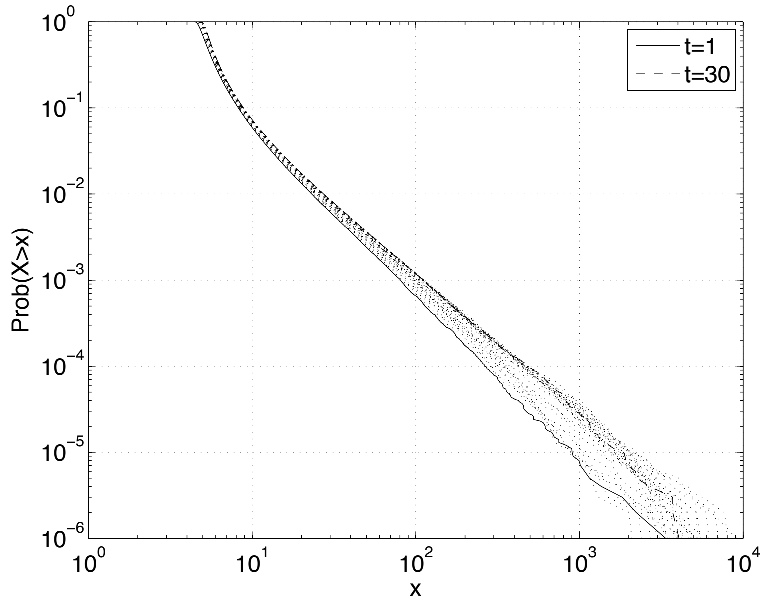

Figure 6: Transition of income distribution after an increase in Var(g)

and $s = 0.3$. Also, $\sigma^2 \equiv \mathrm{Var}(\log \epsilon)$ is set 1.1 and $\mathrm{E}(\epsilon) = 1$. Under the parameter values, the income distribution converges to a Pareto distribution with exponent $\xi = 2$. Then, we increase the wealth growth variance σ^2 by 10% to 1.2. Figure 6 plots the transition of the income distribution after this jump of σ^2 for 30 years. The plot at $t = 1$ shows the initial distribution with $\xi = 2$. We observe that the distribution converges to a flatter one (lower ξ) in less than 10 years. Therefore, the decline and increase of the excess returns that took place in decades as shown in Figure 4 can cause the movement of Pareto exponent in the same time horizon in our Solow model.

Next we consider an effect of progressive income tax. Clearly, taxation has a direct effect on wealth accumulation by lowering the annual increment of wealth as well as the effect through the altered incentives that households face. The impact of income tax legislation in the 1980s has been extensively argued in the context of the recent U.S. inequalization. In Figure 1, we observe an unprecedented magnitude of decline in Pareto exponent right after the Tax Reform Act in 1986, as studied by Feenberg and Poterba [5].

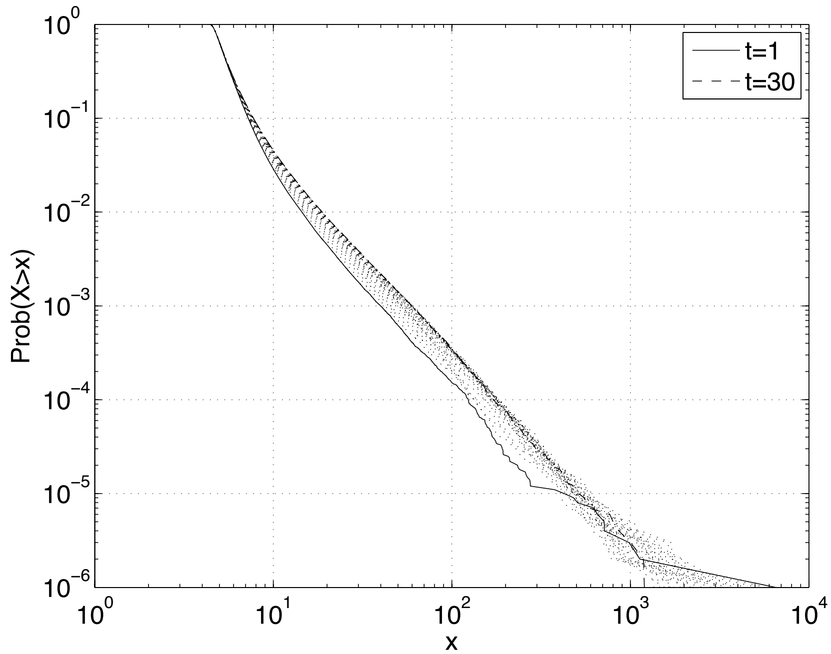

Figure 7: Transition of income distribution after an increase in Var(g)

Although the stable exponent after the downward leap suggests that the sudden decline was partly due to the tax-saving behavior, the steady decline of the Pareto exponent in the 1990s may suggest more persistent effects of the tax act. Piketty and Saez [15] suggests that the imposition of progressive tax around the second world war was the possible cause for the top income share to decline during this period and stay at the low level for a long time until 1980s.

We incorporate a simple progressive tax in our simulated model as follows. We set the marginal tax rate at 50% which is applied for the income that exceeds three times worth of z. Starting from the stationary distribution under the 50% marginal tax, we reduce the tax rate to 20%. The result is shown in Figure 7. We observe the decline of Pareto exponent and its relatively fast convergence. This confirms that the change in progressive tax has an impact on Pareto exponent as well as on the income share of top earners.

Finally, we investigate the impact of the savings rate on Pareto exponent. As our analysis has shown above, the savings rate per se does not affect ξ at the stationary distribution. It turns out that the savings rate does not affect ξ in the transition either. This is because the savings rate affects the returns to wealth through the reduced reinvestment as well as the savings from the labor income. The left panel of Figure 8 shows the case in which the savings rate starts at 0.3 and is reduced by 1% every year. We observe that the Pareto exponent is preserved during the transition.

The savings rate can affect the Pareto exponent if the change in the savings rate are different between the labor income and the capital income. Let us consider a simple "Cambridge" growth model in which the rate of savings from labor income s_L is lower than the saving rate of capital income s_H. Suppose that $s_H = 1$ and $s_L = 0.45$ initially, and suppose that s_L declines by 1 percent every year. The right panel in Figure 8 shows the transition of income distribution. We observe a steady decline in the Pareto exponent, unlike the previous case. This is an interesting observation in the following sense. In a multiplicative process with reflexive lower bound, it is sometimes argued that the movement in the bound does not affect the tail exponent immediately. This is because the tail is usually "too far" from the lower bound to "feel" the change in the lower bound. It seems not the case for us. When ξ is around 2, the mean of the distribution is finite and thus the lower bound can have an immediate impact on the tail exponent. When $\xi = 1$, the lower bound is almost never felt by the tail, and the change in the lower bound causes the change in the tail only in a very long run. When $E(g) \geq 1$, then the lower bound is never felt, the distribution of $x_{i,t}$ fails to converge, and it follows a log-normal distribution with increasing log-variance linearly in time. This is the escaping case of the income distribution evolution as we discuss shortly.

All the variations investigated above show that the Pareto exponent does not decline forever in various environments. This is in fact a general characteristic of the Pareto distribution in the Solow model: ξ must be always greater than one. It suggests that there is a certain limit to an inequalization process. Is there a distinct possibility that this limit is not warranted? Namely, is there a possibility in our framework for an "escaping" inequalization in which the log-variance of income grows without bound?

Sources for such an escaping case are limited. It requires $E(g) \geq 1$, namely, the mean

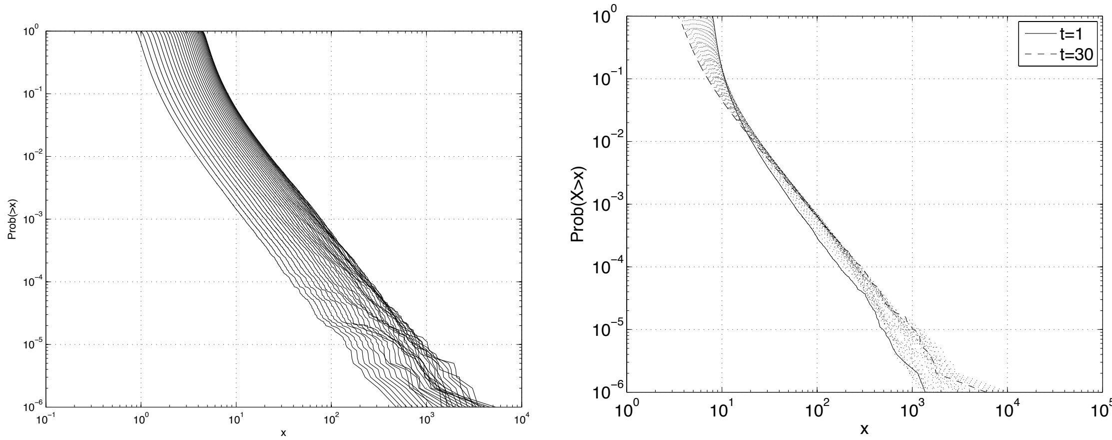

Figure 8: Impact of reduced savings rate on income distribution. Solow model (left) and Cambridge growth model (right)

returns to wealth is greater than the overall growth rate of the economy. This can happen if, say, the investment is heavily subsidized and financed by labor income tax.

The Cambridge growth model studied above provides an interesting alternative case. Consider a simple case $s_H = 1$ and $s_L = 0$ in which the capital income is all reinvested and the labor income is all consumed. Then, the wealth accumulation process (15) becomes a pure multiplicative process! This implies that the income distribution follows a lognormal development, in which the log-variance of income grows linearly in time. Whereas the assumption $s_H = 1$ and $s_L = 0$ is extreme, this seems a possibility worth investigating once we recall that the personal savings rate in the U.S. experiences values below zero recently.

4 Ramsey model

In the last section, we note the possibility that the different savings behavior across the wealth group may have important implications on the Pareto exponent. In order to pursue this issue, we need to depart Solow model and incorporate optimization behavior of households. We work in a Ramsey model in this section. The model is analytically tractable when the savings rate and portfolio decisions are independent of wealth levels. Since Samuelson [17] and Merton [13], we know that this is the case when the utility exhibits a constant relative risk aversion. We draw on a paper by Angeletos [2] in which Aiyagari model [1] is modified to incorporate undiversifiable investment risks. Each consumer is endowed with a unit labor, an initial capital $k_{i,0}$, and a "backyard" production technology that is the same as before (1). The labor endowment is supplied inelastically.

Households can engage in lending and borrowing $b_{i,t}$ at the interest rate R_t. Labor can be hired at wage w_t. The labor contract is contingent on the realization of the technology shock $a_{i,t}$.

A household solves the following maximization problem.

$$\max_{(c_{i,t},b_{i,t},k_{i,t+1},l_{i,t},y_{i,t})} \sum_{t=0}^{\infty} \beta^t \frac{c_{i,t}^{1-\theta}}{1-\theta} \tag{25}$$

subject to:

$$\begin{aligned} c_{i,t} + k_{i,t+1} + b_{i,t+1} &= \pi_{i,t} + (1-\delta)k_{i,t} + R_t b_{i,t} + w_t l_{i,t} \qquad (26) \\ \pi_{i,t} &= y_{i,t} - w_t l_{i,t} \qquad (27) \end{aligned}$$

and the production function (1). At the optimal labor hiring $l_{i,t}$, the profit holds that $\pi_{i,t} = r_{i,t}k_{i,t}$ where $r_{i,t} \equiv (1-\alpha)^{(1-\alpha)/\alpha}(a_{i,t}/w_t)^{(1-\alpha)/\alpha}$.

We define a human wealth H_t as the expected discounted present value of future wage income stream:

$$H_t \equiv \sum_{\tau=t}^{\infty} w_\tau \prod_{s=t+1}^{\tau} R_s^{-1}. \tag{28}$$

Then $H_t = w_t + R_{t+1}^{-1}H_{t+1}$. Then the budget constraint is:

$$c_{i,t} + k_{i,t+1} + b_{i,t+1} + R_{t+1}^{-1}H_{t+1} = W_{i,t} \tag{29}$$

where we define a household wealth $W_{i,t}$ as:

$$W_{i,t} \equiv r_{i,t}k_{i,t} + R_t b_{i,t} + H_t \tag{30}$$

Consider a balanced growth path in which R_t and w_t is constant over time. The household's problem has a recursive form:

$$V(W) = \max_{c,k',b'} c^{1-\theta}/(1-\theta) + \beta E\left[V(W')\right] \tag{31}$$

subject to

$$W = c + k' + b' + R^{-1}H' \tag{32}$$

$$W = rk + Rb + H \tag{33}$$

This dynamic programming allows the following solution:

$$c = (1-s)W \tag{34}$$

$$k' = s\phi W \tag{35}$$

$$b' = s(1-\phi)W - R^{-1}H' \tag{36}$$

and the value function $V(W) = BW^{1-\theta}/(1-\theta)$.

Proof is outlined as follows. The first order conditions and the envelope condition are:

$$c^{-\theta} = \beta E[r'V'(W')] \tag{37}$$

$$c^{-\theta} = \beta R' E[V'(W')] \tag{38}$$

$$V'(W) = c^{-\theta} \tag{39}$$

The guessed functional forms of c, k', b' are consistent with the budget constraint (32). By imposing the guessed functional forms:

$$0 = \beta B(h)E[(r'-R)(W')^{-\theta}] \tag{40}$$

$$(1-s)^{-\theta} = \beta BE[r'(W'/W)^{-\theta}] \tag{41}$$

$$B = (1-s)^{-\theta} \tag{42}$$

Also impose the guess on the definition of wealth:

$$W' = (\phi r' + (1-\phi)R)sW \tag{43}$$

Then the constants are determined by:

$$0 = E[(r'-R)(\phi r' + (1-\phi)R)^{-\theta}] \tag{44}$$

$$s^{\theta} = \beta E[r'(\phi r' + (1-\phi)R)^{-\theta}] \tag{45}$$

$$B = (1-s)^{-\theta} \tag{46}$$

Thus, at the balanced growth path, the household wealth evolves multiplicatively (43). Therefore the wealth W follows a lognormal process. Define a relative wealth $W/E(W)$. Then the relative wealth does not have a stationary distribution. Eventually, a vanishingly small fraction of individuals possesses almost all wealth. This is not consistent with the empirical evidence. The non-human wealth $A \equiv W - H$ follows the dynamic equation $A' = gA + gH - H'$. This is a pure multiplicative process if the human capital grows at the same rate as the non-human capital.

The difference from the Solow model occurs from the fact that the consumption function is linear in income in the Solow model whereas it is linear in wealth in the Ramsey model with CRRA preference. The linear consumption function arises in a quite narrow specification of the model, however, as Carroll and Kimball [3] argues. For example, labor income uncertainty generates a concave consumption function even with complete markets. We can also consider the case in which the household cannot borrow at all. This hinders the households from investing in the risky capital financed by their future wage income. The consumption function becomes convex with respect to wealth, and the saving rate is decreasing in wealth, as shown by Carroll and Kimball [3].

The concavity of the consumption means that the saving rate is high for the households with low wealth. This precautionary savings by the low wealth group serves as the repulsive lower bound of the wealth accumulation process. Then the Pareto exponent of the tail distribution of wealth is determined by the balance of the precautionary savings and the diffusion of the wealth. This can be shown by numerical simulations.

5 Final remarks

This paper demonstrates that a variation of the Solow model is able to generate the Pareto distribution as the stationary distribution of income at the balanced growth path. We explicitly determine the Pareto exponent by the fundamental parameters, and provide an economic interpretation for its determinants.

The model prediction is taken to the historical movement of the Pareto exponents in the U.S. It is shown that the long run movements of excess returns of the risky assets and the progressive tax may have caused the inverted U pattern of the Pareto exponents. It is also shown that the zero or negative personal savings rate may break the condition for the stationary Pareto distribution and cause a steady growth of the cross-section variance of the logarithm of income.

An important determinant of the Pareto exponent is the variance of the returns to wealth, which is difficult to measure. We use the excess returns to the risky assets as the proxy for the variance. However, the link between the two is not firmly established. Also, in order to generate the benchmark Pareto exponent $\xi = 2$, we need to assume a large variance in the productivity shock (σ^2 in the simulations), whereas the benchmark exponent is naturally obtained if we use the personal savings rate and the excess returns. In order to fill in these gaps, it seems that more analysis of the excess returns in the Ramsey model is needed.

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