Bequests, taxation and the distribution of wealth in a general equilibrium model

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This paper examines the role of bequests and of taxation on bequests for the distribution of wealth. We investigate a model with overlapping generations and heterogenous households where parents derive utility directly from their bequests. We obtain all results analytically. Using the coefficient of variation as the measure of inequality, bequests per se diminish the inequality of wealth since they raise private savings and hence average wealth holdings more than the variance of wealth. From a policy perspective, taxing bequests and redistributing government revenue lump-sum among the young generation further decreases wealth inequality.

Keywords: bequest, taxation, wealth inequality, OLG model, analytical solution

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1 Introduction

Most industrial countries levy a tax on wealth transfers. However, there are substantial differences in the legal framework of the tax system. In France and Germany, on the one hand, the tax is levied on inheritances. The institutional setting further forces donors to divide their estate equally among their own children (see Cremer and Pestieau, 2003). In the United States and in the United Kingdom on the other hand, there is a tax on estates and

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donors enjoy more freedom of bequests, although state rules might restrict disinheritance. What many countries seem to have in common is an ongoing and controversial debate about taxation of wealth transfers. Some countries, including the US, contemplate to phase out taxes on wealth transfer in the near future.

One of the main arguments in the public and academic discussion is the role of wealth transfers for the inequality of wealth. Wealth is highly concentrated: in many industrial countries, the share of the richest 1% of households in net worth is estimated to be 20-30% (see Davies and Shorrocks, 2000), whereas an equal distribution would imply that any π% of the population hold π% of the total wealth. Wealth transfers in form of bequests or inter-vivo transfers are often seen as one of the major culprits for the inequality of wealth.[2] Since there is some concern about the level of concentration, the taxation of wealth transfers is frequently identified as an adequate policy to mitigate the concentration of wealth.

This paper investigates the role of bequests for the distribution of wealth and the effects of redistributive taxation.[3] We construct a simple model with stochastic individual income to analyze distributional effects by comparing properties of distributions in an overlapping generations setting. While focusing on steady states for our main results, we provide a complete analysis of transitional dynamics as well.

We find that intergenerational wealth transfers per se have an equalizing effect on the distribution of wealth when the coefficient of variation is chosen as the measure of inequality. Since this result can be seen in general equilibrium only, we consider general equilibrium analysis as being important. In our model, bequests have two effects. On the one hand, there is an increase in the variance of wealth. On the other, as these transfers imply that wealth holdings of a family at a certain point in time are determined not only by own income but by a weighted sum of own and ancestor's income, average wealth holdings increase as well. As the greater average wealth compensates for higher variance, the inequality of wealth, as measured by the coefficient of variation, falls. We further find that this result is robust when correlation across parent's and child's income, endogenous labour supply and a CES instead of a logarithmic specification of the utility function are introduced into the model.

[2] See Charles and Hurst (2003) for an empirical analysis of the reasons for a positive relationship between wealth of parents and children *before* bequests. Bowles and Gintis (2002) discuss the various mechanisms through which economic status is transfered across generations. The effect of tax changes on the importance of gifts relative to bequests are analyzed by Bernheim et al. (2004).

[3] We do not study efficiency aspects as e.g. Blumkin and Sadka (2004) who analyse the efficiency cost of estate taxation. See also Cremer and Pestieau (2001) and Cremer et al. (2003) for an optimal tax analysis under asymmetric information or Grossmann and Poutvaara (2005).

When we turn to economic policy, we allow the government to tax bequests and redistribute revenue among the young. We find that the redistributive policy reduces the variance of wealth while keeping the average wealth holding constant. As a result, inequality of wealth – again measured by the coefficient of variation – falls.

Finally, we analyze how taxation affects the Gini coefficient. Our results are robust to the choice of inequality measure - taxation and redistribution decreases inequality. We are therefore confident that our results can directly be used for policy debates. Note that due to the simplicity of our modeling choice, we are able to derive all results apart from the Gini result in ch. 5.3 analytically.

The relation between intergenerational transfers and the wealth distribution has already found some attention among economists. In contrast to the frequently alleged concentration increasing influence (e.g. Meade, 1976; Wilhelm, 1997), the results of some models indicate that intergenerational wealth transfers imply more equality of wealth. The best known argument is the compensation principle between parents and children where bequests serve to compensate differences in random labour income (Becker and Tomes, 1979; Davies, 1986; Davies and Kuhn, 1991; see Davies and Shorrocks, 2000, ch. 2.3 for a survey[4]).

When it comes to taxation of inheritances, some argue that when "inheritance plays an equalizing role, it seems likely that <...> taxes would be disequalizing" (Davies, 1986, p. 539). In fact, this result has been claimed or shown by Becker and Tomes (1979) or Davies (1986) for a long-run steady state. It was put into some perspective by Davies and Kuhn (1991) who found that in the short-run, taxation could be equalizing. Some seem to summarize the current state of knowledge as if only under exceptional circumstances, taxation of bequests would imply a more equal distribution of wealth: "if there are incomplete markets <...> taxation <...> can reduce <...> inequality" (Gokhale et al., 2001, p. 97).[5]

Compared to this literature, the present paper confirms that bequests per se imply a more equal wealth distribution indeed. The mechanism stressed here, however, is completely independent of the Becker-Tomes compensation principle. The wealth distribution becomes more equal because of the increase in expected wealth which overcompensates the increase in the variance of wealth such that the coefficient of variation falls. As this mechanism relies

[4] There is also a large literature that looks at wealth inequality and bequests from a quantitative perspective. Heer (2001) and Gokhale et al. (2001) are recent examples. They also provide an overview of previous work.

[5] See, however, early work by Pestieau and Posen (1979) who show in a framework with more assets and taxes that taxation can be equalizing.

on capital accumulation, a general equilibrium setup is important.[6]

The reason why we find that taxation of bequests has an equalizing effect lies in the assumption about what parents value. Becker (1974) introduced the concept of "social income" which was used by Becker and Tomes (1979) in the form of "family income" or by Davies (1986) and others (again, see the references in Davies and Shorrocks, 2000, ch. 2.3) as "family wealth". The idea - put simply - is that parents care about total income of children, i.e. labour income plus inheritances after tax and lump-sum transfers. As a consequence, when maximizing utility, parents take family income as the disposable income which they split optimally between own consumption and total income of children. In the present paper, parents care about (after-tax) bequests per se - we therefore follow the joy-of-giving approach. With this setup, parents do not take family income into account but simply their own income which they split optimally between own consumption and bequests. This simple difference in preferences implies that under joy-of-giving, taxation of bequests is neutral with respect to average wealth and therefore, by decreasing the variance of bequests, implies lower inequality in wealth.[7]

The rest of the paper is organized as follows. The next section presents the basic model. Section 3 studies the evolution path of the economy and steady state properties. In section 4, we study the distribution of wealth and the role of bequests for inequality. We also allow for a correlation between parent's and child's income as an extension to the basic model. Section 5 then investigates distributional effects of taxation and checks robustness of our results by using the Gini coefficient as an alternative measure for inequality. The final section concludes.

2 The model

2.1 Households

We consider a society with overlapping generations. Each individual lives for two periods. In each period t there is a large number n of families or dynasties consisting of one parent and one child. When young, individuals work and earn labour income. When old, parents are retired, consume their savings, and leave a bequest to the child. We assume that workers differ with respect to their ability and hence productivity. Let l_{it} denote the effective inelastic

[6]Other general equilibrium models analysing the effect of bequests but not focusing on our questions include e.g. Stiglitz (1969), Ioannides and Sato (1987) or Bhattacharya (1998).

[7]For a survey of transfer motives and some of their implications, see Masson and Pestieau (1997).

labour supply of an individual i. At the beginning of period t, each worker draws l_{it}, where $l_{it} > 0$, from an identical distribution with expectation and variance given by

$$E(l_{it}) = \bar{l} \equiv 1, \quad \mathrm{Var}(l_{it}) = \sigma^2. \tag{1}$$

As the l_{it} are identically and independently distributed (iid) random variables, their covariance is zero, $\mathrm{Cov}(l_{ir}, l_{is}) = 0$, for $r \neq s$. Without loss of generality, we set $\bar{l} = 1$.

Since in each period an equal number of individuals enters and leaves the economy, there is a stationary number of families. While the microeconomic level – that is, individual income, inheritance, savings – is characterized by uncertainty, there is certainty on the per capita level – average efficient labour supply, capital-per-worker, and interest rate and wage are nonrandom variables. For example, the individual efficient labour supply l_{it} is a random variable with variance $\mathrm{Var}(l_{it}) = \sigma^2$. The average efficient labour supply is $\Sigma_{i=1}^{n} l_{it}/n$ with $\mathrm{Var}(\Sigma_{i=1}^{n} l_{it}/n) = \sigma^2/n$ which tends to zero for $n \to \infty$. Hence, for $n \to \infty$, the probability that the average efficient labour supply deviates from its expectation is zero.

Individuals consume in both periods and leave a bequest that immediately passes to their child at the end of the second period. A person belonging to family i, born at the beginning of period t, maximizes lifetime utility

$$U_{it} = U\left(c_{it}^{y}, c_{it+1}^{o}, b_{it+1}\right), \tag{2}$$

by choosing consumption c_{it}^{y} when young, c_{it+1}^{o} when old, and the bequest b_{it+1} passed on to the child. Note that utility depends on the amount b_{it+1} the child receives after tax. This utility function captures the joy-of-giving idea: "consumers leave bequests simply because they obtain utility directly from the bequest" (Abel and Warshawsky, 1988, p. 145).[8] In the first period, the budget constraint for an individual of generation t is

$$w_t l_{it} + b_{it} + g_t = c_{it}^{y} + s_{it}, \tag{3}$$

where b_{it} denotes after tax inheritance received from the parent, $w_t l_{it}$ stochastic income depending on wage w_t per efficiency unit and the random ability of the individual l_{it}, g_t the uniform lump-sum transfer received from the government in case that it levies a tax on bequests, and s_{it} savings. In the second period, the constraint is

$$s_{it}\left[1 + r_{t+1}\right] = c_{it+1}^{o} + (1+\tau)\, b_{it+1}, \tag{4}$$

[8] For an empirical evaluation of various bequest motives, see e.g. Light and McGarry (2003) and the references therein.

where r_{t+1} is the interest rate and $\tau \geq 0$ the tax on bequests. Parents have to take into account that part of their wealth transferred to the child may be collected by the government. With a positive tax rate τ, intending to leave b_{it+1} to the child, the parent has to bequeath $(1+\tau)b_{it+1}$. The individual decision is under certainty, since bequests and labour income are received before deciding about consuming and saving in the first period, and the interest rate is not random.

To keep things as simple as possible, we start with a Cobb-Douglas utility function (see app. 7.2.1 for an extension)

$$U_{it} = \alpha \ln c_{it}^{y} + (1-\alpha)\left[\beta \ln c_{it+1}^{o} + (1-\beta)\ln b_{it+1}\right], \tag{5}$$

with $1/2 < \alpha < 1$, $0 < \beta < 1$. We view second-period consumption and bequests as a composite good and β captures the relative importance of the two. The preference parameter α, capturing intertemporal preferences, must be larger than $1/2$ as otherwise individuals would value future utility from consumption and bequests more than utility from present consumption and the implied time preference rate would be negative.[9] From the first-order conditions one can easily derive consumption in each period, savings, and bequests left to the child

$$\begin{aligned}
c_{it}^{y} &= \alpha\left[w_t l_{it} + b_{it} + g_t\right], && (6)\\
s_{it} &= (1-\alpha)\left(w_t l_{it} + b_{it} + g_t\right), && (7)\\
c_{it+1}^{o} &= \beta s_{it}\left[1 + r_{t+1}\right], && (8)\\
b_{it+1} &= \frac{(1-\beta)s_{it}\left[1 + r_{t+1}\right]}{1+\tau}. && (9)
\end{aligned}$$

Independent of their ability or inherited wealth, individuals always consume the share α of their income $w_t l_{it} + b_{it} + g_t$ in the first period. The rest is saved and yields the return $1 + r_{t+1}$. At the end of the second period, the share β of savings plus the accrued interest is consumed, $1-\beta$ is passed to the child.

Note that savings in equation (7) are independent of τ. The tax on wealth transfers drives a wedge between the relative prices of consumption and leaving a bequest. As a consequence, households will substitute first period consumption for bequests. However, due to the Cobb-Douglas utility, substitution and income effect neutralize each other so that a tax on bequests does not influence individual savings. However, taxing bequests and

[9] We show in the appendix in section 7.2.1 that one of our main results holds also for a more general CES utility function. Section 7.1.1 discusses the specification in (5) and the implied time preference rate. Section 7.1.2 shows that an alternative specification of (5) yields identical results.

redistributing tax revenue will have an effect on households' budgets through g_t and thus on the individual amount saved.

2.2 Firms

Firms use labour and capital as inputs and produce a single (numeraire) good that can be consumed or invested. Labour is supplied inelastically (but see app. 7.2.2). There is perfect competition and firms use a Cobb-Douglas technology. In intensive form, the production function can be expressed as $y_t = f(k_t) = Ak_t^{\gamma}$, where y_t and k_t are output per worker and capital per worker, respectively. In equilibrium, factors are paid their marginal products

$$r_t = \frac{\partial f(k_t)}{\partial k_t} - \delta = \gamma A k_t^{\gamma-1} - \delta, \tag{10}$$

$$w_t = f(k_t) - k_t \frac{\partial f(k_t)}{\partial k_t} = (1-\gamma) A k_t^{\gamma}, \tag{11}$$

where δ is the depreciation rate.

2.3 The government

The only tax instrument of the government is the tax $\tau \geq 0$ on bequests. The government levies this tax and redistributes revenue lump-sum among the young generation. In each period, the government's budget is balanced. Every young individual receives the same transfer g_t that corresponds to the average tax revenue per tax case,

$$ng_t = \tau \Sigma_{i=1}^{n} b_{it}. \tag{12}$$

There is variation in the individual bequests and accordingly different tax revenues per bequest. Due to the large number of families, the average tax revenue of the government is deterministic, however. Hence, the government does not need to form expectations about tax revenue.

3 Transitional dynamics and steady state

We are now in the position to study the dynamics of capital intensity k_t and to calculate its steady state value. Furthermore, we can investigate the development of the distribution of wealth: The law of large numbers tells us that if the individual probability to "draw" a certain amount of wealth equal to at most $\bar{a}$ is given by π%, then, in a large economy, a share of π% of the whole population will hold at most that amount of wealth $\bar{a}$. In order to understand the national distribution of wealth, we therefore just have to understand the properties of the distribution of individual wealth.

In what follows, we investigate convergence in two dimensions: convergence of the capital stock at the macroeconomic level and convergence of the distribution of wealth at the microeconomic level. The distribution of wealth will be studied under the assumption that the economy has already reached the macroeconomic steady state.

3.1 The capital stock

Let us first study the dynamics of k_t. From the goods' market equilibrium it follows that households' assets in period $t+1$ – all owned by members of the generation born in t – equal the period's capital stock. We will show that a (deterministic) stable steady state exists. While such a proof is self-evident in deterministic settings, it is not obvious in our economy with stochastic productivity at the individual level. Using the same approach as for l_{it} mentioned at the beginning of Section 2, we get

$$k_{t+1} = \frac{K_{t+1}/n}{L_{t+1}/n} = \frac{\Sigma_{i=1}^{n} s_{it}/n}{\Sigma_{i=1}^{n} l_{it+1}/n}. \tag{13}$$

Using (7), (9) and (12), observing that for a sufficiently large economy, i.e. for $n \to \infty$, $\Sigma_{i=1}^{n} l_{it}/n = E(l_{it}) = 1$, and substituting further for w_t and r_t from (10) and (11), we obtain after rearranging (see app. A.1)

$$k_{t+1} = c_1 k_t^{\gamma} + c_2 k_t, \tag{14}$$

where $c_1 \equiv (1-\gamma\beta)(1-\alpha)A$ and $c_2 \equiv (1-\alpha)(1-\beta)(1-\delta)$. Equation (14) determines the evolution of k_t starting with an initial value. Note that k_t is independent of the tax rate τ. Taxing wealth transfers *and* redistributing tax yields does not influence the growth path of the economy. (This would not be the case if households had preferences other than Cobb-Douglas.)

3.2 Macroeconomic steady state

To compute the steady state capital intensity, let $k_{t+1} = k_t = \overline{k}$ in (14) and solve for $\overline{k}$

$$\overline{k} = \left(\frac{c_1}{1-c_2}\right)^{(1-\gamma)^{-1}}. \tag{15}$$

The wage $\overline{w}$ and interest rate $\overline{r}$ are calculated from the marginal product of labour and capital in the steady state. Plugging $\overline{k}$ into the factor demand curves (10) and (11), we derive

$$
\begin{aligned}
\overline{r} &= \gamma A \frac{1-c_2}{c_1} - \delta, & (16)\\
\overline{w} &= (1-\gamma) A \left(\frac{c_1}{1-c_2} \right)^{\gamma/(1-\gamma)}. & (17)
\end{aligned}
$$

Since $0 < \frac{dk_{t+1}}{dk_t}\Big|_{\overline{k}} = \gamma + (1-\gamma)c_2 < 1$, the steady state is locally stable. A simple graphical analysis reveals that the steady state is also globally stable.

3.3 The evolution of wealth

We now turn to the evolution at the microeconomic level. Wealth a_{it+1} of family i in period $t+1$ is owned by the parent and consists of the savings of the previous period. Equation (7) implies

$$a_{it+1} = s_{it} = (1-\alpha)(w_t l_{it} + b_{it} + g_t). \tag{18}$$

Substituting for b_{it} from (9) and g_t from (12) gives

$$a_{it+1} = (1-\alpha) w_t l_{it} + \frac{(1-\alpha)(1-\beta)(1+r_t)}{1+\tau} a_{it} + \frac{\tau(1-\alpha)(1-\beta)(1+r_t)}{1+\tau} k_t. \tag{19}$$

This difference equation describes the evolution of wealth holdings of a family. As can be seen from equation (19), several factors determine family wealth in $t+1$. First, wage and interest rate of the previous period, which again depend on the capital intensity of that period. Second, stochastic income and family wealth of period t, which influences current wealth via bequests. And finally, the third term of equation (19), the government transfer the young generation receives.

To calculate an explicit solution for the evolution of wealth a_{it}, we consider an economy which is in its macroeconomic steady state. The capital intensity k, wage w, and interest rate r take their values in (15), (16), and (17) so that we obtain from equation (19)

$$a_{it+1} = c_3 l_{it} + c_4 a_{it} + c_5, \tag{20}$$

where $c_3 \equiv (1-\alpha)\overline{w}$, $c_4 \equiv \frac{(1-\alpha)(1-\beta)(1+\overline{r})}{1+\tau}$ and $c_5 \equiv \frac{\tau(1-\alpha)(1-\beta)(1+\overline{r})}{1+\tau}\overline{k}$. Wealth in period $t+1$ depends on labour market success today (the first term), wealth today (the second) and government transfers (the third term). Note that $\overline{k}$, $\overline{w}$, and $\overline{r}$ might vary with parameter changes since steady states reflect differences in households' willingness to leave wealth to the child or in their saving rate.

When solving equation (20) recursively, we obtain

$$a_{it} = c_5 \Sigma_{s=0}^{t-1} c_4^s + c_4^t a_{i0} + c_3 \Sigma_{s=0}^{t-1} c_4^{t-1-s} l_{is}. \tag{21}$$

Intuitively, one would want the value of c_4 to lie between zero and one as otherwise the sums do not converge. Analytically, it can be shown that this is indeed true (cf. app. A.2). As one can see from equation (21), family wealth in period t depends on the initial wealth a_{i0} of the family at $t = 0$, the transfer of the government, and the stochastic productivities l_{is}, $s = 0, \ldots, t-1$, of all preceding generations. The more luck the ancestors had, the higher was their labour income and the higher are wealth holdings of the current generation, since part of the ancestors' luck is shifted into the future via bequests. However, due to the factor c_4, the influence of distant luck on current wealth is weaker than that of the parent's luck.

3.4 The distribution of wealth

After having studied the actual distribution of wealth, we now look at the characteristics of the distribution of wealth. Being in $t = 0$ today, what is the expected level of wealth at some future point in time t? The expected value of a_{it} is, using (21) and taking (1) into account,

$$E\left(a_{it}\right) = c_5 \Sigma_{s=0}^{t-1} c_4^s + c_4^t a_{i0} + c_3 \Sigma_{s=0}^{t-1} c_4^{t-1-s} = c_4^t a_{i0} + \left(c_3 + c_5\right) \frac{1 - c_4^t}{1 - c_4}. \tag{22}$$

Some algebra shows us (see app. A.3) that $\frac{c_3+c_5}{1-c_4} = \overline{k}$. Hence, we can write this equation as

$$E\left(a_{it}\right) = \left(a_{i0} - \overline{k}\right) c_4^t + \overline{k}.$$

Obviously, expected wealth increases over time when $a_{i0} < \overline{k}$, it decreases when $a_{i0} > \overline{k}$: When households are "rich" (i.e. they have wealth in $t = 0$ above average $\overline{k}$), their wealth tends to decrease, when they are "poor" it tends to increase. All households, independently of their initial value a_{i0}, have the same expected wealth $\overline{k}$ in the long run, i.e. for t approaching infinity,

$$E\left(a_{i\infty}\right) = \frac{c_3 + c_5}{1 - c_4} = \overline{k}. \tag{23}$$

This is due to the fact that shocks are iid such that each household, from whatever level it starts, has the same expected future path of labour productivities. The long-run mean needs to be identical to the aggregate capital stock per worker $\overline{k}$ as the aggregate capital stock is just the sum of individual wealth holdings and as all individuals are equal in this expected sense.

Clearly, when $a_{i0} = \overline{k}$, the expected wealth level for each family at each instant t (and not only in the long run) is $E\left(a_{it}\right) = \overline{k}$. Note that this is not surprising when remembering that we are already in a macroeconomic steady state. As the latter implies that the average capital stock is given by $\overline{k}$ for each t, it must be that the expected wealth holding of a representative family is also $\bar{k}$ for each t. We will base some of our subsequent discussion on the assumption of $a_{i0} = a_0 = \bar{k}$. Our basic difference equation for wealth will then read

$$a_{it} = c_5 \Sigma_{s=0}^{t-1} c_4^s + c_4^t a_0 + c_3 \Sigma_{s=0}^{t-1} c_4^{t-1-s} l_{is}. \tag{24}$$

The variance of the distribution of wealth is, using (21) and employing again (1) and its covariance implication,

$$\text{Var}(a_{it}) = c_3^2 \sigma^2 \Sigma_{s=0}^{t-1} (c_4^{t-1-s})^2 = c_3^2 \sigma^2 \frac{1 - c_4^{2t}}{1 - c_4^2}. \tag{25}$$

The variance unambiguously increases over time but approaches a constant,

$$\text{Var}\left(a_{i\infty}\right) = \frac{c_3^2 \sigma^2}{1 - c_4^2}. \tag{26}$$

When we want to know whether the distribution of a_{it} as a whole, and not just its mean and variance, converges to a limiting distribution, we need to understand whether the sum $\Sigma_{s=0}^{t-1} c_4^{t-1-s} l_{is}$ in (21) converges for $t \to \infty$. While this is clear for the first two terms in (21), this is less obvious for the remaining term given the stochastic nature of individual productivity l_{is}. As $0 < c_4 < 1$ and if we are willing to assume that $\text{Var}(l_{is}) < \infty$, the two-series theorem – a simplified version of the well-known three-series theorem (Shiryaev, 1996, p. 386–387) – implies that $\Sigma_{s=0}^{t-1} c_4^{t-1-s} l_{is}$ converges, for $t \to \infty$, with probability 1.[10] We may conclude that a limiting distribution for a_{it} exists in a fairly general setting.

4 Wealth inequality and bequests

In this section, we will investigate the role of bequests per se for the inequality of wealth within and across generations. We leave the analysis of tax effects for the next section and set $\tau = 0$. Knowing the expectation and the variance of wealth, we use as our measure of inequality the coefficient of variation $CV(a_{it})$,

$$CV(a_{it}) = \frac{\sqrt{\text{Var}(a_{it})}}{E\left(a_{it}\right)}. \tag{27}$$

Using this measure raises at least two questions: Why the coefficient of variation and why the coefficient of variation of wealth? The coefficient of variation is, up to a monotonic transformation, a member of the popular generalized entropy class of inequality measures

[10] Note that the assumption of a finite variance is not satisfied for all commonly used distributional assumptions for l_{is} (e.g. Kleiber and Kotz, 2003). However, the empirical evidence suggests that $\text{Var}(l_{is}) < \infty$ is a realistic assumption.

(see, e.g., Kleiber and Kotz 2003, Ch. 2) and has a geometric interpretation just like the familiar Gini index (Formby, Smith and Zheng 1999). Unlike the Gini coefficient, it is a simple function of the first and second moments of the underlying distribution. This renders it particularly suitable in our analytical framework (as also e.g. Davies and Kuhn, 1991, or Becker and Tomes, 1979) with its emphasis on closed form solutions.

Regarding the second issue, wealth inequality can be argued to be of interest per se. Moreover, it can easily be shown (see app. A.4) that the determinants of our wealth distribution are identical to the determinants of the distribution of utility in our model. Comparing wealth levels therefore allows statements about relative "happiness" levels. The same is true when expected utility or even when dynasties (thinking beyond individuals that live for two periods only) are analyzed. Understanding the distribution of wealth is therefore equivalent to understanding properties of other distributions of interest as well.

4.1 Bequests decrease inequality

Inserting (23) and (26) into (27), we get an expression for the coefficient of variation as a function of β. As the sign of $dCV(a_{i\infty})/d\beta$ remains ambiguous analytically,[11] we obtain information about the effect of bequests on inequality by comparing two economies, A and B. We assume that they are identical, except that in economy B households bequeath wealth (B as bequest), i.e. $0 < \beta < 1$, while in A they do not ($\beta = 1$). In both economies, $c_5 = 0$ as $\tau = 0$ in this section. In economy A, plugging (23) and (26) into (27) and observing that $\beta = 1$ implies $c_4 = 0$, the coefficient of variation is constant for each t,

$$CV(a_{it}^A) = \sigma. \tag{28}$$

This is intuitively clear, considering that without bequests there is no intergenerational link and the only stochastic impact results from own current income. In economy B, the coefficient of variation from equation (27) increases over time as parents bequeath wealth to the child. In a steady state, the coefficient of variation CV is, utilizing again (23) and (26),

$$CV\left(a_{i\infty}^B\right) = \sigma \frac{1 - c_4}{\sqrt{1 - c_4^2}} = \sigma \sqrt{\frac{1 - c_4}{1 + c_4}}. \tag{29}$$

The new determinant in the bequest economy is the factor c_4 whose origin can best be seen from the solution of the household's difference equation for wealth in (21): c_4 is the weight with which past labour productivities affect current wealth. This link between labour

[11] A sufficient (but not necessary) condition under which $dCV(a_{i\infty})/d\beta \geq 0$ is $2 - 1/\alpha \leq \gamma$. This holds for $\gamma \approx 1/3$ and α close to $1/2$, i.e. with relatively patient households.

productivities of previous generations and current wealth exists only because of bequests. With no willingness to bequeath, i.e. $b = 1$, c_4 would be zero.

Since $0 < c_4 < 1$ implies that the fraction in (29) is smaller than one,

$$CV\left(a_{i\infty}^B\right) < CV\left(a_{i\infty}^A\right). \tag{30}$$

Inequality of wealth is lower in the bequest-economy B than in economy A where parents derive no utility from leaving a bequest. In contrast to the intuition and general perception, bequests reduce the intragenerational inequality of wealth.

As mentioned by Davies and Kuhn (1991), bequests may be equalizing through dampening shocks - the Becker-Tomes compensation principle. An equalizing effect is also found in our model. But the reason for it to occur is different. Families in economy B are simply richer on average than families in economy A, because part of the wealth of the parent is transferred to the child. This can formally be seen from $\overline{k}$ in (15) which is decreasing in β. This leads to a rise in the denominator of CV in economy B, compared to economy A.

The same happens to the numerator, however. While capital intensity is higher in economy B, workers' labour earnings in the steady state are also higher in relation to economy A so that the variance of wealth due to income uncertainty – one might call this the life-cycle component of savings – is higher. In addition, uncertainty due to bequests also raises the variance of wealth.

When analyzing the effect of β on the CV in (27) for both numerator and denominator, we find that the rise in expected wealth in the denominator is stronger through bequests than the rise in the standard deviation of wealth in the numerator. Looking at $E\left(a_{i\infty}\right)$ in (23) and $\mathrm{Var}(a_{i\infty})$ in (26), we see that bequests, i.e. $\beta < 1$, affect the labour market channel c_3 and the wealth channel c_4 (see (20)). As c_3 cancels out in the CV, see e.g. (29), only the effect of bequests on c_4 remains. As c_4 is smaller than unity, bequests increase expected wealth more than the standard deviation of wealth. As a consequence, wealth is more equally distributed in an economy with bequests. This line of reasoning generalizes to models, presented in section 7.2, with endogenous labour supply and a CES utility function.

4.2 Bequests with correlation of labour income

Solon (1992) and Zimmermann (1992), studying intergenerational income mobility in the U.S., point out that there is substantial correlation between income of parents and children. Their results indicate that the correlation of sons' log earnings with respect to fathers' incomes is at least 0.4. We therefore now relax the assumption that abilities and hence labour incomes of parent and child are uncorrelated. With positive correlation in earnings, the probability that children of high income parents earn themselves above average labour income is also high. In addition, these children receive relatively large inheritances so that wealth concentration increases. Is it then still true that in economy B, the economy with bequests, wealth is more equally distributed than in economy A?

We restrict ourselves to an analytically tractable form of correlation. Following Davies and Kuhn (1991), we assume that effective labour supply regresses to the mean across generations according to

$$l_{it+1} = \bar{l} + \nu \left[l_{it} - \bar{l} \right] + \epsilon_{it+1},$$

where $\bar{l}$ denotes the expected effective labour supply, which for our purposes is set equal to one, ν with $0 < \nu < 1$ expresses the strength of correlation between fathers' and sons' abilities, and the ϵ_{it+1} represent iid shocks with zero mean, finite variance and a lower bound sufficient to keep $l_{it+1} > 0$. In $t = 0$, the process starts with $l_{i0} = \bar{l} + \epsilon_{i0}$, where $\bar{l} = 1$.

Despite the correlation of income, the economy behaves almost exactly as before: capital intensity still evolves according to equation (14). In economy A where bequests are absent (i.e. $\beta = 1$ and therefore $c_4 = 0$), family wealth a^A_{it+1} from equation (20) is given by (recall that we still assume that the government levies no tax on bequests so that $c_5 = 0$ as well)

$$a^A_{it+1} = (1 - \alpha)\overline{w}^A l_{it} = c^A_3 \left[1 + \Sigma^t_{j=0} \nu^{t-j} \epsilon_{ij} \right],$$

where $c^A_3 = (1-\alpha)\overline{w}^A$. The expected wealth holding is $E\left(a^A_{it+1}\right) = c^A_3$, the variance of wealth is $\mathrm{Var}(a^A_{it+1}) = (c^A_3)^2 \sigma^2 \Sigma^t_{j=0} \nu^{2j}$, and the coefficient of variation is given by

$$CV(a^A_{it+1}) = \sigma \sqrt{\Sigma^t_{j=0} \nu^{2j}} = \sigma \sqrt{\frac{1 - \nu^{2(t+1)}}{1 - \nu^2}}. \tag{31}$$

In contrast to the coefficient of variation (28) under iid abilities, the concentration of wealth increases over time here: Though parents do not bequeath wealth, they do "bequeath their ability". While income correlation leaves the expected value $E\left(a^A_{it+1}\right)$ constant, the variance of income and hence the dispersion of wealth increases. In view of $0 < \nu < 1$, the limit of $CV(a^A_{it})$ is

$$CV\left(a^A_{i\infty}\right) = \sigma \sqrt{\frac{1}{1 - \nu^2}} > \sigma. \tag{32}$$

Hence positive correlation of abilities and thereby income makes the distribution of wealth more unequal.

In the bequest society B, where $0 < \beta < 1$, the wealth accumulating process is more complicated. According to equation (20), setting $\tau = 0$ and thus $c_5 = 0$, we get $a^B_{it} = c^t_4 a_0 +$

$c_3\Sigma_{s=0}^{t-1}c_4^{t-1-s}l_{is}$. The expected value is still easy to calculate and given by $E\left(a_{it}^B\right)=\overline{k}=\frac{c_3}{1-c_4}$. As is shown in app. A.5, the coefficient of variation is, for $t\rightarrow\infty$,

$$CV\left(a_{i\infty}^B\right)=\sigma\sqrt{\frac{1-c_4}{1+c_4}}\sqrt{\frac{1+c_4\nu}{(1-\nu^2)(1-c_4\nu)}}. \tag{33}$$

A comparison with (29) reveals that correlation of income implies a higher inequality of wealth: The term in brackets is larger than unity as the numerator is larger than one and the denominator is smaller than one, given that $0<\{c_4,\nu\}<1$. We conclude that correlated abilities across generations increase wealth inequality both in economies with and without bequests.

Concerning the effect of bequests on equality, equality of wealth is still higher in bequest economies than in no-bequest economies: Comparing (32) and (33) yields

$$\frac{CV\left(a_{i\infty}^B\right)}{CV\left(a_{i\infty}^A\right)}=\frac{\sigma\sqrt{\frac{1-c_4}{1+c_4}}\sqrt{\frac{1+c_4\nu}{(1-\nu^2)(1-c_4\nu)}}}{\sigma\sqrt{\frac{1}{1-\nu^2}}}=\sqrt{\frac{1-c_4}{1+c_4}}\sqrt{\frac{1+c_4\nu}{1-c_4\nu}}. \tag{34}$$

Some algebra reveals that this ratio is less than or equal to 1 if and only if $0\leq(1-c_4)(1-\nu)$. This condition also follows from $0<\{c_4,\nu\}<1$. The result that bequest economies are characterized by lower inequality than economies without bequests is, therefore, robust to the introduction of serially correlated abilities.

4.3 Social mobility

Besides intragenerational inequality, one may also pay attention to other dimensions of inequality as for example social mobility across generations. Social mobility, as used here, is the degree to which child's wealth status may deviate from parent's status. It measures the ability of descendants of poor families to become rich and vice versa. Not surprisingly, in our model bequests have a negative influence on social mobility. While without bequests mobility is perfect, the wealth status of the child being solely determined by his own ability, bequeathing part of their wealth parents also transfer part of their wealth status.

As a formal measure of the degree of immobility, we use the correlation of parent-child wealth (e.g. Conlisk, 1974). With bequests and $\tau=0$, equation (20) implies that family wealth in $t+1$ is given by $a_{it+1}=c_3l_{it}+c_4a_{it}$. The correlation $\mathrm{Cor}(a_{it+1},a_{it})$ of wealth holdings between parent and child is then

$$\mathrm{Cor}(a_{it+1},a_{it})=\frac{\mathrm{Cov}(a_{it+1},a_{it})}{\sqrt{\mathrm{Var}(a_{it+1})\mathrm{Var}(a_{it})}}=\frac{E\left[\left(a_{it+1}-E\left(a_{it+1}\right)\right)\left(a_{it}-E\left(a_{it}\right)\right)\right]}{\sqrt{\mathrm{Var}(a_{it+1})\mathrm{Var}(a_{it})}}.$$

Substituting $a_{it+1} = c_4 a_{it} + c_3 l_{it}$ yields

$$\mathrm{Cor}(a_{it+1}, a_{it}) = \frac{c_4 \mathrm{Var}(a_{it}) + E[c_3 l_{it} a_{it} - c_3 a_{it} - c_3 l_{it} E\left(a_{it}\right) + c_3 E\left(a_{it}\right)]}{\sqrt{\mathrm{Var}(a_{it+1}) \mathrm{Var}(a_{it})}}.$$

Given $\mathrm{Cov}(l_{it}, a_{it}) = 0$ and therefore $E\left(l_{it} a_{it}\right) = E\left(l_{it}\right) E\left(a_{it}\right)$, we obtain

$$\mathrm{Cor}(a_{it+1}, a_{it}) = c_4 \sqrt{\frac{\mathrm{Var}(a_{it})}{\mathrm{Var}(a_{it+1})}} > 0,$$

so that there is indeed correlation between parent's and child's wealth. The strength of correlation depends on the parameter c_4 and the variance of wealth, which from (25) depends on further parameters and on time.

As in the long-run, we obtain $\lim_{t \longrightarrow \infty} \mathrm{Cor}(a_{it+1}, a_{it}) = c_4$, we can easily discuss some determinants of social mobility. Given the definition of c_4 after (20), the interest rate from (16) and with c_1 and c_2 from (14), c_4 equals $(1-\alpha)(1-\beta)(1+\overline{r})/\left(1+\tau\right)$, where $\overline{r} = \gamma \frac{1-(1-\alpha)(1-\beta)(1-\delta)}{(1-\gamma\beta)(1-\alpha)} - \delta$. The parameters of interest here, i.e. those who affect bequests and thereby social mobility directly, are β and τ.

In a no-bequest economy ($\beta = 1$) the interest rate is finite (and given by $\gamma/\left(\left(1-\gamma\right)\left(1-\alpha\right)\right) - \delta$) which implies that $c_4 = 0$. As a consequence, with no bequests, there is zero correlation between wealth in t and $t+1$. Wealth is purely determined by own iid labour income and there is perfect social mobility. In an economy with bequests ($\beta < 1$), a tax τ on bequests reduces c_4 and thereby the link between wealth of subsequent generations. Summarizing, no bequests and a high tax on bequests are beneficial for social mobility.

5 Wealth inequality and taxation

The previous section has shown that parental willingness to bequeath reduces the intragenerational inequality of wealth. We now ask how policy should react to this. Does taxing bequests further decrease inequality? We study this question for an economy without correlation of labour income in the next subsection and with correlation of labour income subsequently. The final subsection provides results on Gini coefficients.

5.1 Taxing bequests decreases inequality

Recall that in our model taxing bequests with a tax rate $\tau > 0$ does not have any influence on private savings on a macroeconomic level: The average capital stock $\overline{k}$ in (15) is constant, no matter if there is a redistributive taxation of bequests or not. This is a crucial consequence of our joy-of-giving approach and contrasts with the Becker-Tomes setup where a tax on bequests reduces average wealth (see e.g. (13) in Davis and Kuhn, 1991, or (15) in Davies, 1986). In our approach, individual consumption and savings in (6) and (7) are a share out of own disposable income, $w_t l_{it} + b_{it} + g_t$. In the Becker-Tomes approach, the decision is based on family income, i.e. own disposable income plus disposable income of the subsequent generation reduced by their inheritance. Decisions based on own disposable income imply neutrality of inheritance tax, decisions based on family income do not. The fundamental reason for this differences is the specification of preferences: In our approach, the utility function has bequests b_{it+1} as one argument, in the Becker-Tomes approach, the argument related to the subsequent generation is their disposable income.

Given our starting point that $a_0 = \overline{k}$, the expectation $E(a_{it})$ of family wealth is $\overline{k}$ for all t, as discussed before (24). Instead of calculating the coefficient of variation, we therefore simply concentrate on the variance of wealth as measure of inequality. Again, we will compare two situations: one, where the government levies no tax ($\tau = 0$) and one with taxation of bequests ($\tau > 0$). We denote the variance as $\text{Var}(a_{it}^{wt})$ in the case where $\tau > 0$ ('with tax') and as $\text{Var}(a_{it}^{nt})$ in the case where $\tau = 0$ ('no tax'). We can calculate $\text{Var}(a_{it}^{wt})$ and $\text{Var}(a_{it}^{nt})$ from equation (25). Note that in the 'no tax' case, we can relate c_4^{nt} to the definition of c_4 after (20), namely $c_4^{nt} = c_4(1+\tau)$. Hence, the ratio is

$$\frac{\text{Var}(a_{it}^{wt})}{\text{Var}(a_{it}^{nt})} = \frac{\Sigma_{s=0}^{t-1} c_4^{2s}}{\Sigma_{s=0}^{t-1} [c_4(1+\tau)]^{2s}}.$$

From $0 < c_4 < 1$ and $\tau > 0$, we know that $c_4 < c_4 [1+\tau]$ and thus for $t \geq 2$

$$\Sigma_{s=0}^{t-1} c_4^{2s} < \Sigma_{s=0}^{t-1} [c_4 [1+\tau]]^{2s}.$$

Hence, the dispersion of family wealth decreases when government levies a tax on bequests and redistributes revenue among the young generation. Therefore, $CV(a_{it}^{wt}) < CV(a_{it}^{nt})$ for all $t \geq 2$ so that the redistributive policy of the government reduces intragenerational inequality.

In the first period, redistribution does not yet work when assuming as we did that wealth is distributed equally in $t = 0$. Hence, bequests of that period are also equally distributed and taxation and redistribution can not alter the concentration of wealth: wealth already is completely equally distributed. But from $t = 2$ on, the tax starts working and wealth is less concentrated subsequently (in an expected sense). Taxation furthermore increases wealth mobility. The higher the tax rate τ, the less parental wealth determines the wealth status of the child. The status of the child is then primarily related to own ability.

The equalizing effect of bequest taxation hinges on several aspects. First, reactions of parents and children depend on the underlying motive for wealth transfers. Second, taxing bequests and redistributing tax revenues does not affect private savings (see app. 7.2.1 for a CES utility function where savings are affected by τ). And third, if wealth was implicitly "lost" in transit, this in turn would have an influence on private savings and the evolution path of the economy with the result that the average wealth holding of families could decrease. Although the redistribution diminishes the dispersion of wealth, lower average wealth holdings could then lead to an increase in inequality.

5.2 Taxing bequests with correlation of labour income

Let us now return to the case of correlated labour income as in ch. 4.2. In contrast to this chapter, however, we now allow for a positive tax rate τ. Given that expected wealth remains invariant to tax changes, we again compute the variance of wealth. For an an economy with bequests we find (see app. A.6)

$$\mathrm{Var}(a_{it}^{B}) = c_3^2\sigma^2 \frac{(1 + c_4 v)}{(1 - c_4^2)(1 - \nu^2)(1 - c_4\nu)}. \tag{35}$$

What is the effect of a change in taxation here?

Note that the special case of the previous subsection shows here as well: With no correlation, i.e. $v = 0$, the variance is given by $c_3^2\sigma^2/\left(1 - c_4^2\right)$. Given that c_3 is independent of τ and c_4 falls in τ, a higher tax decreases the variance of wealth and thereby its coefficient of variation.

In the case of $\nu > 0$, the result can also easily be seen: A smaller c_4, caused by a higher tax rate, decreases the numerator and increases both terms $(1 - c_4^2)$ and $(1 - c_4\nu)$ in the denominator. As a result, the variance and the coefficient of variation of wealth decreases through a higher tax rate. The intuition for this finding is unchanged when compared with the previous subsection.

5.3 The distribution of wealth and the Gini coefficient

Our coefficient of variation results were rather general in that they only required the existence of the first and second moment of a_{it}, thus requiring us solely to specify the expected value and variance of the individual productivities l_{it} in (1). Other commonly used measures of inequality are more difficult to analyze in our setup. For example, a measure of inequality widely used in the public is the amount of wealth owned by the richest $\pi\%$ of the population.

As this amount is generally far higher than π% of total wealth, a society is viewed to have become more equal if this amount reduces. Perfect equality would mean that for any $\pi \in [0, 100]$, π% of the population own π% of the total wealth. The corresponding Gini coefficient (or indeed any reasonable measure of inequality) would then be zero.

Unfortunately, the Gini and related indices require information beyond the lower-order moments. Nevertheless, in order to obtain some insights on the robustness of our results to the choice of inequality measure, we now study further details on the exact distribution of wealth - making stronger assumptions than in preceding sections. We then briefly summarize some simulation results for the Gini coefficient.

We start from equation (24). As stochastic labour productivities l_{is} are iid, we may replace $c_3\Sigma_{s=0}^{t-1}c_4^{t-1-s}l_{is}$ by $\Sigma_{s=1}^{t}c_3c_4^{s-1}l_{is} \equiv \Sigma_{s=1}^{t}X_s$ (since both objects have identical distributions) and consider

$$a_{it} = c_5\Sigma_{s=0}^{t-1}c_4^s + c_4^t a_0 + \Sigma_{s=1}^{t}X_s. \tag{36}$$

For any fixed t, the first two terms amount to a (non-stochastic) shift to the right, while the remaining term represents the sum of independent but not identically distributed random variables. Exact distributions of such quantities are, in general, not easily available. A simple closed-form solution for $\Sigma_{s=1}^{t}X_s$ would exist if we assumed the l_{is} to be normally distributed; however, this would imply the possibility of negative productivities (and thereby wages) which is not plausible. More generally, distributions of sums are available if one is willing to assume, like Ioannides and Sato (1987), that earned income follows a (non-normal) stable distribution. Unfortunately, stable distributions create further problems: The normal distribution is the only stable distribution admitting a finite variance. Hence, the coefficient of variation is not meaningful for non-normal stable laws. Infinite variances are also not well supported by the data.

It would therefore be desirable to choose a distribution not possessing these drawbacks while remaining reasonably tractable. A recent survey of models for the size distribution of income is provided by Kleiber and Kotz (2003). One plausible candidate for the distribution of productivities l_{is} is the two-parameter gamma distribution, suggested by Salem and Mount (1974) among others, with density $f(l_{it}) = \frac{1}{\lambda^{\phi}\Gamma(\phi)}l_{it}^{\phi-1}e^{-l_{it}/\lambda}$, parameters $\lambda > 0$ and $\phi > 0$, support $[0, \infty)$ and $\Gamma(\phi)$ representing the gamma function, $\Gamma(\phi) = \int_0^{\infty} t^{\phi-1}e^{-t}dt$. In short, we assume $l_{is} \sim \mathrm{Ga}(\phi, \lambda)$. Recall that for $Y = cX$ the density of Y is given by $f_Y(y) = f_X(y/c)c^{-1}$. Then, in our case, the density of X_s is $g(x_s) = \frac{1}{(c_3c_4^{s-1}\lambda)^{\phi}\Gamma(\phi)}x_s^{\phi-1}e^{-x_s/\left(c_3c_4^{s-1}\lambda\right)}$. Thus

$$X_s = c_3c_4^{s-1}l_{is} \sim \mathrm{Ga}\left(\phi, c_3c_4^{s-1}\lambda\right) \equiv \mathrm{Ga}(\phi, \beta_s). \tag{37}$$

A representation obtained by Moschopoulos (1985), see App. 7.3 for further details, provides the density g of $\Sigma_{s=1}^{t} X_s$ in the form

$$g(y) = c_4^{(t-1)t\phi/2} \sum_{k=0}^{\infty} \delta_k \frac{y^{t\phi+k-1} e^{-y/\tilde{\beta}}}{\Gamma(t\phi+k)\tilde{\beta}^{t\phi+k}}, \qquad y > 0, \tag{38}$$

where $\tilde{\beta} = c_3 \lambda c_4^{t-1}$, $\delta_0 = 1$ and

$$\delta_{k+1} = \frac{\phi}{k+1} \Sigma_{i=1}^{k+1} \delta_{k+1-i} \Sigma_{s=1}^{t-1} \left(1 - c_4^s\right)^i, \quad k = 0, 1, 2, \dots .$$

Recalling that, for $Z = Y + d$, the density of Z is given by $f_Z(z) = f_Y(z - d)$, the density of a_{it}, see (36), is finally given by $f_{a_{it}}(z) = g(z - d)$, where $d = c_5 \Sigma_{s=0}^{t-1} c_4^s + c_4^t a_0$.

This expression, although fairly complex when compared to textbook densities, is quite convenient for computational purposes and allows, among other things, to evaluate and plot the exact density of the distribution of wealth for any t. (Alternative specifications of the distribution of the l_{is} generally lead to even more involved objects.)

In order to obtain some insights for the Gini coefficient we conducted a limited simulation study on the basis of (38). Recall that the expectation of a gamma random variable X is given by $E(X) = \phi\lambda$. An empirically plausible value for the shape parameter ϕ is 3, in order to have $E(l_{is}) = 1$ as required by (1) we chose $\lambda = 1/3$. We set the remaining model parameters to $\alpha = 0.7$, $\beta = 0.7$, $\gamma = 0.3$, $\delta = 0.8$ and $A = 2$ and considered an economy comprising $n = 10,000$ agents, initially after $t = 20$ periods; this was replicated 1,000 times. Our results suggest that the Gini coefficient, like the coefficient of variation, is decreasing in τ, where we employed $\tau = 0.1(0.1)0.5$. Simulations for longer time horizons of $t = 50$ and $t = 100$ confirmed these findings. It would seem that our results on the inequality of wealth are not confined to the coefficient of variation, although alternative measures are considerably harder to study analytically.

6 Conclusion

This paper analyzed the impact of bequests on the distribution of wealth assuming a joy-of-giving motive. We distinguish between the effect of bequests per se and taxation of bequests. Bequests per se are captured by a preference parameter $(1 - \beta)$ which measures the "utility-elasticity of bequests", i.e. the willingness of parents to bequeath. The higher the parents willingness to give, the higher is the share of parental disposable income the child receives. The paper shows that both expected wealth and the variance of wealth go up with higher joy-of-giving. Since the effect on expected wealth dominates, the coefficient of variation also goes down: More bequests make the distribution of wealth more equal. The reason for this result is the effect bequests have on capital accumulation which we took into account in our general equilibrium analysis.

From a policy perspective, by levying a wealth transfer tax and redistributing revenue among the young generation, the government can further reduce the concentration of wealth. The higher the tax τ on bequests, the lower is the variance of wealth, while average wealth holdings are not affected. As a consequence, the coefficient of variation is reduced by the tax. Hence, the government can follow a bequest taxation policy in order to reduce wealth inequality, even though bequests per se - the willingness to bequeath - already result in lower wealth inequality. We find this inequality-reducing effect of taxation (which would also be found in unintended-bequest setups) due to our assumption of a joy-of-giving motive which removes Becker-Tomes type "family wealth" considerations.

While these results hold for the coefficient of variation as a measure of inequality, simulation suggests that they also hold for other, "more popular" measures like the Gini coefficient. Taxing bequests reduces not only the coefficient of variation but also the Gini coefficient.

The appendix to this paper analyses various extensions - CES utility and endogenous labour supply. We show that the first result - bequests per se increase equality due to capital accumulation - is robust to these extensions. Future work could check whether the taxation result also survives under these more general specifications.

7 Appendix

7.1 On the instantaneous utility function

The instantaneous utility function (5) models second-period consumption and bequests as a composite good. The parameter α captures intertemporal preferences, the parameter β intratemporal preferences within period 2. The implication for the time preference rate of this formulation is worked out in sect. 7.1.1. The subsequent section studies whether our central results survive for an alternative formulation of the utility function.[12]

7.1.1 The time preference rate

This appendix derives the time preference rate (TPR) for the utility function (5). Given our proposal of a definition for the TPR, it shows that the time preference rate is a function only

[12] We are grateful to one Referee for a very constructive discussion of these issues.

of α. The parameter β affects only the decision of how to split second period expenditures between consumption and bequests. Surprisingly, this issue has not explicitly been discussed in the literature before.

The time preference rate can be defined as the marginal rate of substitution between c_t and c_{t+1} at identical consumption levels $c_t = c_{t+1}$ (see e.g. Buiter, 1981, p. 773),

$$1 + TPR \equiv \left. \frac{\partial U_{it}/\partial c_{it}^{y}}{\partial U_{it}/\partial c_{it+1}^{o}} \right|_{c_t = c_{t+1}} . \tag{39}$$

Applying this to (5) gives

$$TPR = \left. \frac{\alpha / c_{it}^{y}}{(1-\alpha)\,\beta / c_{it+1}^{o}} \right|_{c_t = c_{t+1}} - 1 = \frac{\alpha}{(1-\alpha)\,\beta} - 1. \tag{40}$$

A change in β indeed changes the intertemporal elasticity of substitution. An alternative definition of the time preference rate which yields identical results is $1+TPR \equiv \frac{\partial U_{it}/\partial\left(\ln c_{it}^{y}\right)}{\partial U_{it}/\partial\left(\ln c_{it+1}^{o}\right)}$: the TPR is the marginal rate of substitution with respect to instantaneous utilities, i.e. $\ln c_{it}^{y}$ and $\ln c_{it+1}^{o}$. This also yields (40).

If we view the TPR, however, as reflecting discounting between present and future points in time with respect to utility levels and thereby view second period expenditure to consist of expenditure on consumption and bequests, an appropriate definition of the time preference rate would be

$$1 + TPR \equiv \frac{\partial U_{it}/\partial \left(\ln c_{it}^{y}\right)}{\partial U_{it}/\partial \left(\left[\beta \ln c_{it+1}^{o} + (1-\beta) \ln b_{it+1}\right]\right)}. \tag{41}$$

Applying this to (5) would give

$$TPR = \frac{\alpha}{1-\alpha} - 1. \tag{42}$$

We believe that the definition in (41) is more appropriate in our context as the narrow definition with respect to consumption levels in (39) neglects that in period two there is expenditure also for bequests (which could be viewed as a second consumption good).

Any change in β we undertake later in the paper therefore is a change about preference within period 2. It is not a change in intertemporal preferences. This can also be seen from (6) and (7) where consumption and saving in the first period depends on α only and not on β. [13]

7.1.2 An alternative utility function

In the literature (e.g. Cremer and Pestieau, 2003), households with a bequest motive are sometimes described by a utility function of the form

$$\tilde{U}_{it} = \ln c_{it}^{y} + \rho \ln c_{it+1}^{o} + \zeta \ln b_{it+1}. \tag{43}$$

[13] We are grateful to Larry Epstein for some brief but very helpful remarks.

The parameter ρ captures the weight attached to second period consumption relative to first period consumption - basically a measure of impatience or time preferences. The parameter ζ measures the importance attached to bequests. We want to understand here whether our main results based on more or less willingness to bequeath from ch. 4 would hold also in a setup where ζ captures the willingness to bequeath and not $1-\beta$ as in our specification in (5).

Preferences in (43) imply consumption and saving rules which are identical to (6) - (9) if α and β are such that

$$\alpha = 1/\left(1+\rho+\zeta\right), \quad \beta = \rho/\left(\rho+\zeta\right) \Leftrightarrow \tag{44a}$$

$$\rho = \frac{\beta(1-\alpha)}{\alpha}, \quad \zeta = \frac{(1-\alpha)(1-\beta)}{\alpha}. \tag{44b}$$

Hence, all results which are built on comparative static analyses where α and β do not change (e.g. the results in ch. 5 on tax effects) remain valid in our setup. If, however, changes in ζ in the alternative formulation in (43) are used to model changes in the willingness to bequeath, both α and β change and we need to see whether our comparative static result from ch. 4.1 - bequests decrease inequality - survives.

As before in ch. 4.1, we consider a no-bequest economy A and a bequest economy B. With the specification in (43), a no-bequest economy A results from $\zeta = 0$ and a bequest economy has a $\zeta > 0$. Expressing (27) for our notation of $\tilde{U}_{it}$, i.e. replacing all α and β by (44a), we find for (28) and (29)

$$CV(a_{it}^{A}) = \sigma, \quad CV(a_{i\infty}^{B}) = \sigma\sqrt{\frac{1-\tilde{c}_4}{1+\tilde{c}_4}},$$

where $\tilde{c}_4 = \frac{(\rho+\zeta)\zeta(1+\bar{r})}{(1+\rho+\zeta)(\rho+\zeta)(1+\tau)}$, $\bar{r}$ is given by $\bar{r} = \gamma A\frac{1-c_2}{c_1} - \delta$ from (16) and constants c_i are given after (14).

Now note that even though we express all variables in terms of ζ, the properties of the variables remain unchanged. By (44b), the constant $\tilde{c}_4$ is identical to c_4, $\tilde{c}_4 = c_4$ and we therefore know that $0 < \tilde{c}_4 < 1$. As a consequence, as before, we know that $CV(a_{it}^{A}) = \sigma > CV(a_{i\infty}^{B})$ - more bequests lead to a lower level of inequality.

Other results in ch. 4.1 such as the effect of correlation of income across generations also continue to hold when willingness to bequeath is captured by ζ. Due to the one-to-one relationship between α, β and γ, ρ, the introduction of the correlation coefficient ν in ch. 4.2 does not affect α or β. Hence, given a certain ζ, more correlation leads to a higher level of inequality also for the alternative formulation in (43). Bequest economies therefore remain less unequal than economies without bequests.

Summarizing, the results in the main part of this paper are robust to the specification of the utility function. Whether we choose the original formulation in (5) or the alternative one in (43), bequests decrease inequality and taxation of bequests decreases inequality even further.

7.2 Two extensions

The model we present can relatively easily be extended in two directions: We can allow for CES utility functions and for endogenous labour supply. Our primary findings are confirmed and closed form solutions still exist.

7.2.1 CES utility

Consider a household whose utility function is given by a CES formulation for period one and two and Cobb-Douglas for the choice within period two,

$$U_t = \alpha \left[c_{it}^y\right]^{1-\theta} + (1-\alpha)\left[\left(c_{it+1}^o\right)^{\beta}\left(b_{it+1}\right)^{1-\beta}\right]^{1-\theta}, \qquad 0 < \theta < 1. \tag{45}$$

Optimal second period behaviour splits savings between consumption and bequests as before, see (8) and (9),

$$c_{it+1}^o = \beta s_{it}\left[1 + r_{t+1}\right], \qquad b_{it+1} = \frac{(1-\beta)s_{it}\left[1 + r_{t+1}\right]}{1+\tau}. \tag{46}$$

Inserting the budget constraint from period one and this solution into (45) gives

$$\begin{aligned} U_t &= \alpha\left[w_t l_{it} + b_{it} + g_t - s_{it}\right]^{1-\theta} \\ &\quad + (1-\alpha)\left[\left(\beta s_{it}\left[1+r_{t+1}\right]\right)^{\beta}\left(\frac{(1-\beta)s_{it}\left[1+r_{t+1}\right]}{1+\tau}\right)^{1-\beta}\right]^{1-\theta} \\ &\equiv \alpha\left[B_t - s_{it}\right]^{1-\theta} + (1-\alpha)\,\Phi s_{it}^{1-\theta}, \end{aligned} \tag{47}$$

where we defined

$$B_t \equiv w_t l_{it} + b_{it} + g_t, \qquad \Phi \equiv \left[\beta^{\beta}\left(\frac{1-\beta}{1+\tau}\right)^{1-\beta}\left[1+r_{t+1}\right]\right]^{1-\theta}.$$

Maximizing (47) with respect to s_{it} yields $\alpha\left[1-\theta\right]\left[B_t - s_{it}\right]^{-\theta} = (1-\alpha)(1-\theta)\,\Phi s_{it}^{-\theta} \Leftrightarrow B_t - s_{it} = \left(\frac{(1-\alpha)\Phi}{\alpha}\right)^{-1/\theta} s_{it}$ and, solved for s_{it} and with $\varepsilon = 1/\theta$,

$$s_{it} = \frac{B_t}{1 + \left\{\frac{\alpha}{(1-\alpha)\Phi}\right\}^{\varepsilon}}. \tag{48}$$

In order to understand the evolution of wealth, we write in analogy to (18)

$$a_{it+1} = s_{it} = \frac{1}{1+\left\{\frac{\alpha}{(1-\alpha)\Phi}\right\}^{\varepsilon}}\left(w_t l_{it} + b_{it} + g_t\right). \tag{49}$$

Substituting as before for b_{it} and g_t from (9) and (12) with $s_{it-1} = a_{it}$ gives an equation corresponding to (19),

$$a_{it+1} = \frac{1}{1+\left\{\frac{\alpha}{(1-\alpha)\Phi}\right\}^{\varepsilon}} w_t l_{it} + \frac{(1-\beta)(1+r_t)}{(1+\left\{\frac{\alpha}{(1-\alpha)\Phi}\right\}^{\varepsilon})(1+\tau)} a_{it} + \frac{(1-\beta)\tau(1+r_t)}{(1+\left\{\frac{\alpha}{(1-\alpha)\Phi}\right\}^{\varepsilon})(1+\tau)} k_t.$$

Also in analogy to the Cobb-Douglas case, we analyze properties of a_{it} by assuming that the economy is in a macroeconomic steady state. The equation replacing (20) then becomes

$$a_{it+1} = c_6 l_{it} + c_7 a_{it} + c_8, \tag{50}$$

where

$$\begin{aligned} c_6 &= \frac{1}{1+\left\{\frac{\alpha}{(1-\alpha)\Phi}\right\}^{\varepsilon}}\overline{w}, \qquad c_7 = \frac{(1-\beta)(1+\overline{r})}{\left(1+\left\{\frac{\alpha}{(1-\alpha)\Phi}\right\}^{\varepsilon}\right)(1+\tau)}, \\ c_8 &= \frac{(1-\beta)\tau(1+\overline{r})}{\left(1+\left\{\frac{\alpha}{(1-\alpha)\Phi}\right\}^{\varepsilon}\right)(1+\tau)}. \end{aligned} \tag{51}$$

Solving the equation recursively, we obtain an equation corresponding to (21),

$$a_{it} = c_8 \Sigma_{s=0}^{t-1} c_7^s + c_7^t a_{i0} + c_6 \Sigma_{s=0}^{t-1} c_7^{t-1-s} l_{is}.$$

In order to be able to say more about a_{it}, we would have to determine whether $c_7 < 1$. While this is analytically cumbersome, we know from a different line of reasoning that it needs to be smaller than unity indeed. Assume it is larger than one and remember that the economy is in a macroeconomic steady state with a constant capital stock. With $c_7 > 1$, the average wealth holding of families would increase. As this is a contradiction to a macroeconomic steady state, we conclude that $c_7 < 1$ indeed. Then, corresponding to (22), the expected value of a_{it} becomes

$$E\left(a_{it}\right) = c_7^t a_{i0} + (c_6 + c_8)\frac{1-c_7^t}{1-c_7} \underset{t\to\infty}{\longrightarrow} \frac{c_6+c_8}{1-c_7}, \tag{52}$$

and its variance is

$$\mathrm{Var}(a_{it}) = c_6^2\sigma^2\frac{1-c_7^{2t}}{1-c_7^2} \underset{t\to\infty}{\longrightarrow} \frac{c_6^2\sigma^2}{1-c_7^2}. \tag{53}$$

We can now proceed as in section 4.1 and compare wealth inequality of two economies in steady state, one where parents bequeath part of their wealth and one where they do not.

Assume that in economy A parents consume their savings in the second period ($\beta = 1$) while in economy B part of wealth is left to the child. In economy B, assuming that there is no tax on bequests levied (therefore $c_8 = 0$), the coefficient of variation is from (53) and (52) - given the definition in (27) -

$$CV\left(a_{i\infty}^B\right) = \frac{\sqrt{c_6^2\sigma^2/\left(1-c_7^2\right)}}{c_6/\left(1-c_7\right)} = \sqrt{\frac{1-c_7}{1+c_7}}\sigma.$$

In economy A, $c_7 = 0$ due to $\beta = 1$ and the coefficient of variation is

$$CV\left(a_{i\infty}^A\right) = \sigma.$$

Thus, for $0 < c_7 < 1$ we have $CV\left(a_{i\infty}^B\right) < CV\left(a_{i\infty}^A\right)$. Again, in the bequest-economy B wealth inequality is lower.

7.2.2 Endogenous labour supply

Consider an extension of our utility function (2) which includes a labour-leisure choice. Instead of $U_{it} = U\left(c_{it}^y, c_{it+1}^o, b_{it+1}\right)$, we have $U_{it} = U(c_{it}^y, \Lambda_{it}, c_{it+1}^o, b_{it+1})$, where Λ_{it} is the share of time used for leisure. The budget constraint when young needs to be modified to read

$$w_t l_{it}\left[1-\Lambda_{it}\right] + b_{it} + g_t = c_{it}^y + s_{it}. \tag{54}$$

The budget constraint of the old remains as in (4). As in (5), we use a Cobb-Douglas type utility function augmented to capture the additional first period trade-off, letting ξ capture the preference for leisure,

$$U_{it} = \alpha\left[\ln c_{it}^y + \xi\ln\Lambda_{it}\right] + (1-\alpha)\left[\beta\ln c_{it+1}^o + (1-\beta)\ln b_{it+1}\right].$$

Again, optimal second period behaviour splits savings between consumption and bequests as in (8), (9) or (46). The utility function therefore becomes

$$U_{it} = \alpha\left[\ln c_{it}^y + \xi\ln\Lambda_{it}\right] + (1-\alpha)\left[\ln\left(\left(\beta\left[1+r_{t+1}\right]\right)^\beta\left(\frac{(1-\beta)\left[1+r_{t+1}\right]}{1+\tau}\right)^{1-\beta}\right) + \ln s_{it}\right].$$

Defining $\Psi \equiv \ln\left(\left(\beta\left[1+r_{t+1}\right]\right)^\beta\left(\frac{(1-\beta)[1+r_{t+1}]}{1+\tau}\right)^{1-\beta}\right)$ and inserting the first period budget constraint (54) provides a well-defined maximization problem where s_{it} and Λ_{it} are to be chosen,

$$U_{it} = \alpha\left[\ln\left(w_t l_{it}\left[1-\Lambda_{it}\right] + b_{it} + g_t - s_{it}\right) + \xi\ln\Lambda_{it}\right] + (1-\alpha)\left[\Psi + \ln s_{it}\right].$$

The first order condition for s_{it} is

$$\begin{aligned} \frac{\alpha}{w_t l_{it}\left[1-\Lambda_{it}\right]+b_{it}+g_t-s_{it}} &= \frac{1-\alpha}{s_{it}} \Leftrightarrow \alpha s_{it}=(1-\alpha)\left[w_t l_{it}\left[1-\Lambda_{it}\right]+b_{it}+g_t-s_{it}\right] \\ &\Leftrightarrow \quad s_{it}=(1-\alpha)\left[w_t l_{it}\left[1-\Lambda_{it}\right]+b_{it}+g_t\right], \end{aligned} \tag{55}$$

an expression that looks very familiar. The first order condition for leisure reads

$$\frac{w_t l_{it}}{w_t l_{it}\left[1-\Lambda_{it}\right]+b_{it}+g_t-s_{it}}=\frac{\xi}{\Lambda_{it}} \Leftrightarrow w_t l_{it}\Lambda_{it}=\xi\alpha\left[w_t l_{it}\left[1-\Lambda_{it}\right]+b_{it}+g_t\right],$$

where the second step inserted savings and rearranged. Solving for leisure gives

$$\begin{aligned} w_t l_{it}\Lambda_{it} &= \xi\alpha\left[w_t l_{it}+b_{it}+g_t\right]-\xi\alpha w_t l_{it}\Lambda_{it} \Leftrightarrow (1+\xi\alpha)\, w_t l_{it}\Lambda_{it}=\xi\alpha\left[w_t l_{it}+b_{it}+g_t\right] \\ &\Leftrightarrow \quad \Lambda_{it}=\frac{\xi\alpha}{1+\xi\alpha}\left[1+\frac{b_{it}+g_t}{w_t l_{it}}\right]. \end{aligned} \tag{56}$$

This solution has familiar properties as well. If there was no non-labour income, Cobb-Douglas preferences would imply constant labour supply, $\Lambda_{it}=\xi\alpha/(1+\xi\alpha)$. Given the presence of bequests and government income, a higher wage rate w_t and higher individual productivity l_{it} imply that the percentage Λ_{it} of time spent as leisure decreases and labour supply increases.

Inserting (56) into the savings expression (55) gives

$$\begin{aligned} s_{it} &= (1-\alpha)\left[w_t l_{it}\left[1-\frac{\xi\alpha}{1+\xi\alpha}\left[1+\frac{b_{it}+g_t}{w_t l_{it}}\right]\right]+b_{it}+g_t\right] \\ &= (1-\alpha)\left[w_t l_{it}-\frac{\xi\alpha}{1+\xi\alpha}\left[w_t l_{it}+b_{it}+g_t\right]+b_{it}+g_t\right] \\ &= (1-\alpha)\left(1-\frac{\xi\alpha}{1+\xi\alpha}\right)\left[w_t l_{it}+b_{it}+g_t\right]=\frac{1-\alpha}{1+\xi\alpha}\left[w_t l_{it}+b_{it}+g_t\right]. \end{aligned}$$

As in the main text in (18) and in the first extension in (49), this allows us to express wealth a_{it+1} as a linear difference equation which provides information about the various moments and other distributional measures of wealth,

$$a_{it+1}=s_{it}=\frac{1-\alpha}{1+\xi\alpha}\left[w_t l_{it}+b_{it}+g_t\right].$$

Substituting as before for b_{it} and g_t from (9) and (12) with $s_{it-1}=a_{it}$ and defining

$$c_9=\frac{1-\alpha}{1+\xi\alpha}\bar{w}, \qquad c_{10}=\frac{1-\alpha}{1+\xi\alpha}\frac{(1-\beta)(1+\bar{r})}{1+\tau}$$

gives the equation corresponding to (20) or (50),

$$a_{it+1}=c_9 l_{it}+c_{10}a_{it}+\frac{1-\alpha}{1+\xi\alpha}\bar{g}.$$

Being interested only in the effect of bequests (leaving other aspects for future research), we set taxes equal to zero. This simplifies this equation to $a_{it+1} = c_9 l_{it} + c_{10} a_{it}$. Solving recursively gives $a_{it} = c_{10}^t a_{i0} + c_9 \Sigma_{s=0}^{t-1} c_{10}^{t-1-s} l_{is}$ and, as in (52), we find $E(a_{it}) = c_{10}^t a_{i0} + c_9 \frac{1-c_{10}^t}{1-c_{10}} \underset{t\to\infty}{\longrightarrow} \frac{c_9}{1-c_{10}}$ and the variance becomes $\mathrm{Var}(a_{it}) \underset{t\to\infty}{\longrightarrow} \frac{c_9^2\sigma^2}{1-c_{10}^2}$. Hence, the coefficient of variation in the bequest economy B is $CV\left(a_{i\infty}^B\right) = \sqrt{\frac{c_9^2\sigma^2}{1-c_{10}^2}} / \frac{c_9}{1-c_{10}} = \sigma\sqrt{\frac{1-c_{10}}{1+c_{10}}}$. In the non-bequest economy A, where $\beta = 1$ and therefore $c_{10} = 0$, we have $CV\left(a_{i\infty}^A\right) = \sigma$. As $c_{10} < 1$, the bequest economy has again lower inequality.

7.3 The sum of independent gamma random variables

The quantity of interest is $\Sigma_{s=1}^t X_s$, i.e. the distribution of the sum of independent but not identically distributed gamma random variables. There are numerous representations of this object in the statistical literature, for our purposes a result due to Moschopoulos (1985) is the most convenient.

Lemma 1 *(Moschopoulos, 1985) Consider $Y \equiv \Sigma_{s=1}^t X_s$ where the X_s are independent* $\mathrm{Ga}(\phi_s, \beta_s)$. *The density g of Y may be written in the form*

$$g(y) = C\Sigma_{k=0}^{\infty}\delta_k \frac{y^{\tilde{\phi}+k-1}e^{-y/\tilde{\beta}}}{\Gamma(\tilde{\phi}+k)\tilde{\beta}^{\tilde{\phi}+k}}, \qquad y > 0,$$

where $\tilde{\phi} \equiv \sum_{s=1}^t \phi_s$, $\tilde{\beta} \equiv \min_s\{\beta_s\}$, $C = \Pi_{s=1}^t \left(\tilde{\beta}/\beta_s\right)^{\phi_s}$,

$$\delta_{k+1} = \frac{1}{k+1}\Sigma_{i=1}^{k+1} i\gamma_i\delta_{k+1-i}, \quad k = 0, 1, 2, ...$$

$\delta_0 = 1$, *and* $\gamma_i = \Sigma_{s=1}^t \phi_s \left(1 - \tilde{\beta}/\beta_s\right)^i / i$.

Note that, apart from the weight δ_k, the kth term in the series, $y^{\tilde{\phi}+k-1}e^{-y/\tilde{\beta}}/\{\Gamma(\tilde{\phi}+k)\tilde{\beta}^{\tilde{\phi}+k}\}$, is the density of a $\mathrm{Gamma}(\tilde{\phi}+k, \tilde{\beta})$ variable.

Applying this lemma to our case (37), we have $\phi_s \equiv \phi$ for all s, hence $\tilde{\phi} = t\phi$, and $\tilde{\beta} \equiv \min_s \beta_s = c_3\lambda c_4^{t-1}$ (since $0 < c_4 < 1$). This implies $\tilde{\beta}/\beta_s = c_4^{t-s}$, yielding $\gamma_i = \phi\sum_{s=1}^t \left(1 - c_4^{t-s}\right)^i / i = \phi\sum_{s=1}^{t-1}\left(1 - c_4^s\right)^i / i$. The coefficient δ_{k+1} is therefore given by

$$\begin{aligned} \delta_{k+1} &= \frac{\phi}{k+1}\Sigma_{i=1}^{k+1}\Sigma_{s=1}^{t-1}\left(1 - c_4^s\right)^i \delta_{k+1-i} \\ &= \frac{\phi}{k+1}\Sigma_{i=1}^{k+1}\delta_{k+1-i}\Sigma_{s=1}^{t-1}\left(1 - c_4^s\right)^i, \quad k = 0, 1, 2, ... \end{aligned}$$

Further, $C = \prod_{s=1}^t c_4^{(t-s)\phi} = \left(c_4^{\sum_{s=1}^{t-1} s}\right)^{\phi} = c_4^{(t-1)t\phi/2}$. This is the representation used in the text.

An additional appendix (referred to in the text by app. A.x) is available at www.waelde.com/publications

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A Referees' Appendix

for "Bequests, taxation and the distribution of wealth in a general equilibrium model" by Martin Bossmann, Christian Kleiber and Klaus Wälde (available at www.waelde.com/publications), forthcoming Journal of Public Economics

A.1 Deriving (14)

We start from (13). Observe that $\Sigma_{i=1}^{n} s_{it}$ from (7) is simply $\Sigma_{i=1}^{n} s_{it} = (1-\alpha)\left(w_t \Sigma_{i=1}^{n} l_{it} + \Sigma_{i=1}^{n} b_{it} + \Sigma_{i=1}^{n} g_t\right)$. As $\Sigma_{i=1}^{n} g_t = n g_t = \tau \Sigma_{i=1}^{n} b_{it}$ from (12), we find

$$\begin{aligned} \Sigma_{i=1}^{n} s_{it} &= (1-\alpha)\left(w_t \Sigma_{i=1}^{n} l_{it} + (1+\tau)\, \Sigma_{i=1}^{n} b_{it}\right) \\ &= (1-\alpha)\left(w_t \Sigma_{i=1}^{n} l_{it} + (1-\beta)(1+r_t)\, \Sigma_{i=1}^{n} s_{it-1}\right), \end{aligned}$$

where the last step inserted (9). Inserting into (13) yields

$$\begin{aligned} k_{t+1} &= \frac{(1-\alpha)\left(w_t \Sigma_{i=1}^{n} l_{it}/n + (1-\beta)(1+r_t)\, \Sigma_{i=1}^{n} s_{it-1}/n\right)}{\Sigma_{i=1}^{n} l_{it+1}/n} \\ &= (1-\alpha)\left(w_t \frac{\Sigma_{i=1}^{n} l_{it}/n}{\Sigma_{i=1}^{n} l_{it+1}/n} + (1-\beta)(1+r_t)\frac{\Sigma_{i=1}^{n} s_{it-1}/n}{\Sigma_{i=1}^{n} l_{it+1}/n}\right) \\ &= (1-\alpha)\left(w_t \frac{\Sigma_{i=1}^{n} l_{it}/n}{\Sigma_{i=1}^{n} l_{it+1}/n} + (1-\beta)(1+r_t)\, k_t \frac{\Sigma_{i=1}^{n} l_{it}/n}{\Sigma_{i=1}^{n} l_{it+1}/n}\right), \end{aligned}$$

where the last step used (13). For a sufficiently large economy, i.e. $n \to \infty$, $\Sigma_{i=1}^{n} l_{it}/n = E(l_{it}) = 1$. Hence, $\frac{\Sigma_{i=1}^{n} l_{it}/n}{\Sigma_{i=1}^{n} l_{it+1}/n}$ approaches unity and we obtain

$$k_{t+1} = (1-\alpha)w_t + (1-\alpha)(1-\beta)(1+r_t)k_t.$$

Substituting further for w_t and r_t from (10) and (11), we obtain

$$\begin{aligned} k_{t+1} &= (1-\alpha)(1-\gamma)\, A k_t^{\gamma} + (1-\alpha)(1-\beta)\left(1+\gamma A k_t^{\gamma-1} - \delta\right) k_t \\ &= (1-\alpha)(1-\gamma)\, A k_t^{\gamma} + (1-\alpha)(1-\beta)\left(1 k_t + \gamma A k_t^{\gamma} - \delta k_t\right) \\ &= \left[1-\gamma+(1-\beta)\gamma\right](1-\alpha)\, A k_t^{\gamma} + (1-\alpha)(1-\beta)(1-\delta)\, k_t \\ &= \left[1-\beta\gamma\right](1-\alpha)\, A k_t^{\gamma} + (1-\alpha)(1-\beta)(1-\delta)\, k_t. \end{aligned}$$

Defining c_1 and c_2, we obtain (14).

A.2 Proof that $0 < c_4 < 1$

Define, for the purposes of this appendix only,

$$\Delta \equiv (1-\alpha)(1-\beta)(1+\bar{r}). \tag{57}$$

We first show that $0 < \Delta \leq 1$. Obviously, Δ is positive. Noting that Δ is decreasing in $\delta \in [0,1]$ and increasing in $\gamma \in [0,1]$, we show that it is smaller than unity as follows

$$\begin{aligned}
\Delta &= (1-\alpha)(1-\beta)\left(1+\gamma\frac{1-(1-\alpha)(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)+(1-\alpha)(1-\beta)\gamma}-\delta\right) \\
&\leq (1-\alpha)(1-\beta)\left(1+\gamma\frac{1-(1-\alpha)(1-\beta)}{(1-\alpha)(1-\gamma)+(1-\alpha)(1-\beta)\gamma}\right) \\
&\leq (1-\alpha)(1-\beta)\left(1+\frac{1-(1-\alpha)(1-\beta)}{(1-\alpha)(1-\beta)}\right)=1.
\end{aligned}$$

Hence $c_4 = \Delta/(1+\tau)$, as defined after (20), is also positive and smaller than unity.

A.3 Proof of $\frac{c_3+c_5}{1-c_4} = \overline{k}$

A.3.1 An auxiliary result

We show here that $(1-\Delta)\,\bar{k} = c_3$. Write this equation as $1-\Delta = c_3/\overline{k}$. Inserting $\overline{r}$ from (16), the LHS can be written as

$$\begin{aligned}
1-\Delta &= 1-(1-\alpha)(1-\beta)(1+\overline{r}) \\
&= 1-(1-\alpha)(1-\beta)\left(1+\gamma A\frac{1-(1-\alpha)(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)A+(1-\alpha)(1-\beta)\gamma A}-\delta\right)
\end{aligned}$$

Rearranging gives

$$\begin{aligned}
1-\Delta &= 1-(1-\beta)\frac{(1-\alpha)(1-\gamma)+\gamma-(1-\alpha)(1-\gamma)\delta}{1-\gamma+(1-\beta)\gamma} \\
&= \frac{1-\gamma-(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}{1-\gamma+(1-\beta)\gamma}.
\end{aligned}$$

Inserting $\overline{k}$ from equation (15) and substituting $\overline{w}$ from equation (17) into $c_3 = (1-\alpha)\overline{w}$, the RHS can be written as

$$\begin{aligned}
\frac{c_3}{\overline{k}} &= \frac{(1-\alpha)(1-\gamma)A\left(\frac{(1-\alpha)(1-\gamma)A+(1-\alpha)(1-\beta)\gamma A}{1-(1-\alpha)(1-\beta)(1-\delta)}\right)^{\frac{\gamma}{1-\gamma}}}{\left(\frac{(1-\alpha)(1-\gamma)A+(1-\alpha)(1-\beta)\gamma A}{1-(1-\alpha)(1-\beta)(1-\delta)}\right)^{\frac{1}{1-\gamma}}} \\
&= \frac{1-\gamma-(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}{1-\gamma+(1-\beta)\gamma}.
\end{aligned}$$

As the expressions from the LHS and RHS are identical, we have shown that $1-\Delta = \frac{c_3}{\overline{k}}$.

A.3.2 The proof

From the definition of c_4 and c_5 (see after (20)), we know that they can be written with the definition of Δ in (57) as $c_4 = \Delta/\left(1+\tau\right)$ and $c_5 = \frac{\tau}{1+\tau}\Delta\overline{k}$. Hence,

$$\frac{c_3+c_5}{1-c_4}=\frac{c_3+\frac{\tau}{1+\tau}\Delta\bar{k}}{1-\frac{\Delta}{1+\tau}}=\frac{c_3+\frac{\tau}{1+\tau}\left(1-\frac{c_3}{\bar{k}}\right)\bar{k}}{1-\frac{1-\frac{c_3}{\bar{k}}}{1+\tau}},$$

where the second equality used $(1-\Delta)\,\bar{k}=c_3$ from A.3.1. Rearranging yields

$$\frac{c_3+\frac{\tau}{1+\tau}\bar{k}-c_3\frac{\tau}{1+\tau}}{\frac{1+\tau+\frac{c_3}{\bar{k}}-1}{1+\tau}}=\frac{c_3(1+\tau)-c_3\tau+\tau\bar{k}}{\frac{\tau\bar{k}+c_3}{\bar{k}}}=\frac{c_3+\tau\bar{k}}{c_3+\tau\bar{k}}\bar{k}=\bar{k}.$$

A.4 Inequality of what?

Measures of (in)equality can be computed for many variables. The conceptually most important quantity for measuring (in)equality across individuals is probably utility U_{it} from (5). Computing U_{it} by inserting optimal consumption and bequest from (6) to (9) gives

$$\begin{aligned}U_{it} &= \alpha\ln\alpha+\alpha\ln\left[w_t l_{it}+b_{it}+g_t\right] \\ &\quad +(1-\alpha)\left[\begin{array}{c}\beta\ln\beta\,(1-\alpha)\,[1+r_{t+1}]+\beta\ln\left(w_t l_{it}+b_{it}+g_t\right) \\ +(1-\beta)\ln\frac{(1-\beta)(1-\alpha)[1+r_{t+1}]}{1+\tau}+(1-\beta)\ln\left[w_t l_{it}+b_{it}+g_t\right]\end{array}\right] \\ &= \alpha\ln\alpha+(1-\alpha)\left[\beta\ln\beta\,(1-\alpha)\,[1+r_{t+1}]+(1-\beta)\ln\frac{(1-\beta)\,(1-\alpha)\,[1+r_{t+1}]}{1+\tau}\right] \\ &\quad +\ln\left[w_t l_{it}+b_{it}+g_t\right].\end{aligned}$$

As parameters and the interest rate are identical across individuals, the source of heterogeneity is period-one income, $w_t l_{it}+b_{it}+g_t$. Given the identity (up to the constant factor $1-\alpha$) between wealth a_{it+1} and period one income from (18), our measure of inequality not only captures wealth inequality per se but also inequality in well-being.

What about expected utility? Heterogeneity of expected utility of individuals before revelation of individual productivities is driven by bequests. This can most easily be seen when looking at a monotone transformation of U_{it}, capturing identical preferences as U_{it} itself, $\tilde{U}_{it}=e^{U_{it}}$. Expected utility is then

$$E\tilde{U}_{it}=\tilde{U}_0 E\left(w_t l_{it}+b_{it}+g_t\right)=\tilde{U}_0\left[w_t\bar{l}+b_{it}+g_t\right].$$

As bequests b_{it} by (9) and the definition of wealth in (18) are related to wealth according to

$$b_{it+1}=\frac{(1-\beta)\,[1+r_{t+1}]}{1+\tau}a_{it+1},$$

expected utility is driven by the same source of heterogeneity as wealth itself. Hence, whether we analyze the distribution of wealth, the distribution of utility or the distribution of expected utility, the fundamental source is always the same. We therefore believe that wealth inequality is a meaningful distributional measure.

One could finally think about dynasties. One would have to construct a measure of utility of a dynasty (by summing up discounted U_{it}s) and look at its distributional properties. Dynasty welfare will probably depend on initial assets a_{jt} of a dynasty j. This conjecture follows from dynamic programming considerations. The value function, being defined by $V(.) = \max_{\{\text{controls}\}} U_{jt}$, captures utility from optimal behaviour. The value function has as arguments the state variables of some system. As the state variable of a dynasty will be its wealth a_{jt} (plus other variable which will not differ across dynasties, however), we believe that even when looking at dynasties, the distribution of wealth will be a crucial determinant for the distribution of welfare.

A.5 Derivation of equation (33)

A.5.1 Auxiliary results

We first present auxiliary results related to the mean reverting process $l_{it+1} = \bar{l} + \nu(l_{it} - \bar{l}) + \epsilon_{it+1}$ introduced in section 4.2. Given $l_{i0} = \bar{l} + \epsilon_{i0}$, we obtain $l_{it+1} = \bar{l} + \Sigma_{j=0}^{t+1}\nu^{t+1-j}\epsilon_{ij}$ with expectation $E(l_{it+1}) = \bar{l}$ and variance $\text{Var}(l_{it+1}) = \nu^2\text{Var}(l_{it}) + \sigma^2 = \sigma^2\Sigma_{j=0}^{t+1}\nu^{2j}$. The covariance $\text{Cov}(l_{it+k}, l_{it})$ between the ability of the child of family i in period $t+k$ and the ability of the ancestor working in period t is defined as

$$\text{Cov}(l_{it+k}, l_{it}) = E[(l_{it+k} - E(l_{it+k}))(l_{it} - E(l_{it}))].$$

Inserting $l_{it+k} = \bar{l} + \nu^k(l_{it} - \bar{l}) + \Sigma_{j=1}^{k}\nu^{k-j}\epsilon_{it+j}$ and taking into account that ϵ_{it+m} and l_{it} are independent for $m > 0$ and thus $E(\epsilon_{it+m}l_{it}) = E(\epsilon_{it+m})E(l_{it}) = 0$ for $m > 0$ we obtain

$$\text{Cov}(l_{it+k}, l_{it}) = \nu^k E[(l_{it} - E[l_{it}])^2] = \nu^k\text{Var}(l_{it}).$$

A.5.2 A first result on the CV

Observe that from (20) with $\tau = 0$ wealth for periods 1 to 4 is given by

$$\begin{aligned}
a_{i1}^B &= c_4 a_0^B + c_3 l_{i0}, \qquad (58)\\
a_{i2}^B &= c_4 a_{i1}^B + c_3 l_{i1} = c_4^2 a_0^B + c_4 c_3 l_{i0} + c_3 l_{i1},\\
a_{i3}^B &= c_4 a_{i2}^B + c_3 l_{i2} = c_4^3 a_0^B + c_4^2 c_3 l_{i0} + c_4 c_3 l_{i1} + c_3 l_{i2},\\
a_{i4}^B &= c_4 a_{i3}^B + c_3 l_{i3} = c_4^4 a_0^B + c_4^3 c_3 l_{i0} + c_4^2 c_3 l_{i1} + c_4 c_3 l_{i2} + c_3 l_{i3}.
\end{aligned}$$

Using the results from section A.5.1, computing the variance yields

$$\begin{aligned}
\mathrm{Var}(a_{i1}^B) &= c_3^2\sigma^2, \\
\mathrm{Var}(a_{i2}^B) &= c_4^2c_3^2\sigma^2 + c_3^2\sigma^2(1+\nu^2) + 2c_4c_3^2\mathrm{Cov}(l_{i1}, l_{i0}) \\
&= c_4^2c_3^2\sigma^2 + c_3^2\sigma^2(1+\nu^2) + 2c_4c_3^2\nu\sigma^2 = \sigma^2c_3^2(1+(c_4+\nu)^2), \\
\mathrm{Var}(a_{i3}^B) &= c_3^2\sigma^2(1+(c_4+\nu)^2+(c_4^2+c_4\nu+\nu^2)^2),
\end{aligned}$$

for periods 1 to 3 and

$$\begin{aligned}
\mathrm{Var}(a_{i4}^B) &= c_4^6c_3^2\sigma^2 + c_4^4c_3^2\mathrm{Var}(l_{i1}) + c_4^2c_3^2\mathrm{Var}(l_{i2}) + c_3^2\mathrm{Var}(l_{i3}) + 2c_4^5c_3^2\mathrm{Cov}(l_{i1}, l_{i0}) \\
&\quad +2c_4^4c_3^2\mathrm{Cov}(l_{i2}, l_{i0}) + 2c_4^3c_3^2\mathrm{Cov}(l_{i3}, l_{i0}) + 2c_4^3c_3^2\mathrm{Cov}(l_{i2}, l_{i1}) \\
&\quad +2c_4^2c_3^2\mathrm{Cov}(l_{i3}, l_{i1}) + 2c_4c_3^2\mathrm{Cov}(l_{i3}, l_{i2}) \\
&= c_3^2\sigma^2(1+(c_4+\nu)^2+(c_4^2+c_4\nu+\nu^2)^2+(c_4^3+c_4^2\nu+c_4\nu^2+\nu^3)^2),
\end{aligned}$$

for period 4. Generalizing this last expression gives

$$\mathrm{Var}(a_{it}^B) = c_3^2\sigma^2\Sigma_{k=1}^t\left(\Sigma_{s=0}^{k-1}c_4^{k-1-s}\nu^s\right)^2. \tag{59}$$

With $E\left(a_{it}^B\right) = \overline{k} = \frac{c_3}{1-c_4}$ the coefficient of variation becomes

$$CV(a_{it}^B) = \sigma(1-c_4)\sqrt{\Sigma_{k=1}^t\left(\Sigma_{s=0}^{k-1}c_4^{k-1-s}\nu^s\right)^2}. \tag{60}$$

A.5.3 Two cases

We now have to distinguish the case of $c_4 \neq \nu$ from $c_4 = \nu$.

Case 1: For $c_4 \neq \nu$, we use the identity

$$\Sigma_{s=0}^{k-1}c_4^{k-1-s}\nu^s = \frac{c_4^k-\nu^k}{c_4-\nu} \tag{61}$$

This yields

$$\begin{aligned}
CV(a_{it}^B) &= \sigma(1-c_4)\sqrt{\frac{1}{(c_4-\nu)^2}\Sigma_{k=1}^t\left(c_4^{2k}+\nu^{2k}-2(c_4\nu)^k\right)} && (62) \\
&= \sigma\frac{1-c_4}{c_4-\nu}\sqrt{c_4^2\frac{1-c_4^{2t}}{1-c_4^2}+\nu^2\frac{1-\nu^{2t}}{1-\nu^2}-2c_4\nu\frac{1-(c_4\nu)^t}{1-c_4\nu}}. && (63)
\end{aligned}$$

For $t \to \infty$, we obtain

$$CV\left(a_{i\infty}^B\right) = \sigma\frac{1-c_4}{c_4-\nu}\sqrt{\frac{c_4^2}{1-c_4^2}+\frac{\nu^2}{1-\nu^2}-\frac{2c_4\nu}{1-c_4\nu}}.$$

Now observe that

$$\begin{aligned}\frac{c_4^2}{1-c_4^2}+\frac{\nu^2}{1-\nu^2}-\frac{2c_4\nu}{1-c_4\nu} &= \frac{c_4^2}{1-c_4^2}+1+\frac{\nu^2}{1-\nu^2}+1-\frac{2c_4\nu}{1-c_4\nu}-2\\ &= \frac{1}{1-c_4^2}+\frac{1}{1-\nu^2}-\frac{2}{1-c_4\nu}\end{aligned}$$

and that

$$\begin{aligned}&\frac{1}{1-c_4^2}+\frac{1}{1-\nu^2}-\frac{2}{1-c_4\nu}\\ =\ &\frac{(1-\nu^2)(1-c_4\nu)+(1-c_4^2)(1-c_4\nu)-2(1-c_4^2)(1-\nu^2)}{(1-c_4^2)(1-\nu^2)(1-c_4\nu)}\\ =\ &\frac{1-c_4\nu-\nu^2+c_4\nu^3+1-c_4\nu-c_4^2+c_4^3\nu-2+2\nu^2+2c_4^2-2c_4^2\nu^2}{(1-c_4^2)(1-\nu^2)(1-c_4\nu)}\\ =\ &\frac{c_4^2-2c_4\nu+\nu^2+c_4^3\nu+c_4\nu^3-2c_4^2\nu^2}{(1-c_4^2)(1-\nu^2)(1-c_4\nu)}=\frac{(c_4-\nu)^2+c_4\nu(c_4^2+\nu^2-2c_4\nu)}{(1-c_4^2)(1-\nu^2)(1-c_4\nu)}\\ =\ &\frac{(c_4-\nu)^2(1+c_4\nu)}{(1-c_4^2)(1-\nu^2)(1-c_4\nu)}.\end{aligned}$$

Hence,

$$\begin{aligned}CV\left(a_{i\infty}^B\right) &= \sigma\frac{1-c_4}{c_4-\nu}\sqrt{\frac{(c_4-\nu)^2(1+c_4\nu)}{(1-c_4^2)(1-\nu^2)(1-c_4\nu)}}\\ &= \sigma\sqrt{\frac{1-c_4}{1+c_4}}\sqrt{\frac{1+c_4\nu}{(1-\nu^2)(1-c_4\nu)}}.\end{aligned} \tag{64}$$

which is (33).

Case 2: For the special case where $c_4=\nu$, $\Sigma_{s=0}^{k-1}c_4^{k-1-s}\nu^s=k\nu^{k-1}$ and equation (60) can be written as

$$CV(a_{it}^B)=\sigma(1-\nu)\sqrt{\Sigma_{k=1}^t k^2\nu^{2k-2}}. \tag{65}$$

We will now show that

$$\Sigma_{k=1}^t k^2\nu^{2k-2}=2\frac{1-\nu^{2t}}{(1-\nu^2)^3}-\frac{1+(2t-1)\nu^{2t}}{(1-\nu^2)^2}-\frac{t^2\nu^{2t}}{1-\nu^2}.$$

Proof. Define $q\equiv v^2$ so that

$$\Sigma_{k=1}^t k^2q^{k-1}=1+2^2q+3^2q^2+4^2q^3+\ldots+(t-1)^2q^{t-2}+t^2q^{t-1}. \tag{66}$$

Multiplying (66) with q we obtain

$$q\Sigma_{k=1}^t k^2q^{k-1}=q+2^2q^2+3^2q^3+4^2q^4+\ldots+(t-1)^2q^{t-1}+t^2q^t. \tag{67}$$

Subtracting (67) from (66) we get

$$(1-q)\Sigma_{k=1}^t k^2q^{k-1}=1+3q+5q^2+7q^3+\ldots+(2t-3)q^{t-2}+(2t-1)q^{t-1}-t^2q^t.$$

Since

$$\Sigma_{k=1}^{t}(2k-1)q^{k-1} = 1 + 3q + 5q^2 + 7q^3 + \ldots + (2t-3)q^{t-2} + (2t-1)q^{t-1},$$

or multiplying with q again

$$q\Sigma_{k=1}^{t}(2k-1)q^{k-1} = q + 3q^2 + 5q^3 + 7q^4 + \ldots + (2t-3)q^{t-1} + (2t-1)q^{t},$$

in order to calculate the difference between both equations

$$\begin{aligned}(1-q)\Sigma_{k=1}^{t}(2k-1)q^{k-1} &= 1 + 2q + 2q^2 + 2q^3 + \ldots + 2q^{t-1} - (2t-1)q^t \\ &= 2 + 2q + 2q^2 + 2q^3 + \ldots + 2q^{t-1} - 1 - (2t-1)q^t \\ &= 2\Sigma_{m=1}^{t}q^{m-1} - 1 - (2t-1)q^t = 2\frac{1-q^t}{1-q} - 1 - (2t-1)q^t,\end{aligned}$$

so that we obtain

$$\Sigma_{k=1}^{t}(2k-1)q^{k-1} = 2\frac{1-q^t}{(1-q)^2} - \frac{1+(2t-1)q^t}{1-q}. \tag{68}$$

Plugging the result from (68) into $(1-q)\Sigma_{k=1}^{t}k^2q^{k-1}$ we obtain

$$(1-q)\Sigma_{k=1}^{t}k^2q^{k-1} = 2\frac{1-q^t}{(1-q)^2} - \frac{1+(2t-1)q^t}{1-q} - t^2q^t,$$

and thus

$$\Sigma_{k=1}^{t}k^2q^{k-1} = 2\frac{1-q^t}{(1-q)^3} - \frac{1+(2t-1)q^t}{(1-q)^2} - \frac{t^2q^t}{1-q}.$$

Substituting ν^2 for q again, we finally have

$$\Sigma_{k=1}^{t}k^2\nu^{2k-2} = 2\frac{1-\nu^{2t}}{(1-\nu^2)^3} - \frac{1+(2t-1)\nu^{2t}}{(1-\nu^2)^2} - \frac{t^2\nu^{2t}}{1-\nu^2}. \tag{69}$$

■

Since $0 < \nu < 1$, the sum from (69) converges for $t \to \infty$ to its limit

$$\Sigma_{k=1}^{\infty}k^2\nu^{2k-2} = \frac{2}{(1-\nu^2)^3} - \frac{1}{(1-\nu^2)^2} = \frac{1+\nu^2}{(1-\nu^2)^3}.$$

As a consequence, $CV(a_{it}^B) = \sigma(1-\nu)\sqrt{\Sigma_{k=1}^{t}k^2\nu^{2k-2}}$ in (65) tends to, for $t \to \infty$,

$$CV\left(a_{i\infty}^B\right) = \sigma(1-\nu)\sqrt{\frac{1+\nu^2}{(1-\nu^2)^3}}. \tag{70}$$

Comparing (32) and (70), we get

$$\frac{CV\left(a_{i\infty}^B\right)}{CV\left(a_{i\infty}^A\right)} = \frac{\sigma\left(1-\nu\right)\sqrt{\frac{1+\nu^2}{(1-\nu^2)^3}}}{\sigma\sqrt{\frac{1}{1-\nu^2}}} = (1-\nu)\sqrt{\frac{1+\nu^2}{(1-\nu^2)^2}} = \frac{\sqrt{1+\nu^2}}{1+v}. \tag{71}$$

Equations (70) and (71) follow from (33) and (34) for $c_4 = \nu$. Given that (70) and (71) are special cases, the economic interpretation of (33) and (34) would therefore be valid here as well, despite the fact that the RHS of (61) is not defined for $c_4 = \nu$ and therefore had to be computed along different lines here.

A.6 Deriving (35)

Assume that labour income is correlated as in ch. 4.2. In contrast to this chapter, allow for a positive tax rate τ. We can then compute the variance of wealth in a bequest economy as in ch. A.5.2: While the expression for wealth in (58) needs to read $a_{i1}^B = c_4 a_0^B + c_3 l_{i0} + c_5$ as $c_5 > 0$ with a positive tax rate, and while the same must hold true for a_{i2}^B, a_{i3}^B and a_{i4}^B, adding this constant c_5 does not affect the expressions for the variances. As a consequence, (59), reproduced here,

$$\mathrm{Var}(a_{it}^B) = c_3^2 \sigma^2 \Sigma_{k=1}^t \left(\Sigma_{s=0}^{k-1} c_4^{k-1-s} \nu^s \right)^2$$

holds for $\tau > 0$ as well.

Now assume $c_4 \neq \nu$ and use the identity $\Sigma_{s=0}^{k-1} c_4^{k-1-s} \nu^s = \frac{c_4^k - \nu^k}{c_4 - \nu}$ from (61). This yields

$$\mathrm{Var}(a_{it}^B) = c_3^2 \sigma^2 \Sigma_{k=1}^t \left(\frac{c_4^k - \nu^k}{c_4 - \nu} \right)^2 = \frac{c_3^2 \sigma^2}{(c_4 - \nu)^2} \Sigma_{k=1}^t \left(c_4^{2k} + \nu^{2k} - 2 \left(c_4 \nu \right)^k \right).$$

Following the same steps as from (62) to (64), we find

$$\mathrm{Var}(a_{it}^B) = \frac{c_3^2 \sigma^2}{(c_4 - \nu)^2} \frac{(c_4 - \nu)^2 (1 + c_4 \nu)}{(1 - c_4^2)(1 - \nu^2)(1 - c_4 \nu)} = c_3^2 \sigma^2 \frac{(1 + c_4 \nu)}{(1 - c_4^2)(1 - \nu^2)(1 - c_4 \nu)}.$$