Das House-Kapital: A Long Term Housing & Macro Model*

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Abstract

There are, by now, several long term, time series data sets on important housing & macro variables, such as land prices, house prices, and the housing wealth-to-income ratio. However, when it comes to the long term evolution, an appropriate and consistent theory that can be employed to think about this data and associated research questions has still been lacking. We present a new housing & macro model that is designed specifically to analyze the long term. The proposed model replicates, with remarkable accuracy, the historical evolution of housing wealth (relative to income) after World War II and suggests a further considerable increase in the future. The model also accounts for the close connection of house prices to land prices in the data.

Key words: Housing Wealth; Non-Residential Wealth; Wealth-to-Income Ratios; House Prices; Land Prices.

JEL classification: E10, E20, O40.

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1 Introduction

Research on housing and macroeconomics has prospered over the last decade.[1] One outcome of this research effort is the increasing availability of long term, time series data, such as data on house prices and land prices (Knoll, Schularick and Steger, 2016), data on rents for dwellings (Knoll, 2016), and data on housing wealth (Piketty and Zucman, 2014a). The importance of this new data is widely discussed in the literature (Bonnet et al., 2014; Davis, Fisher and Whited, 2014; Gennaioli, Shleifer and Vishny, 2014; Jórda, Schularick and Taylor, 2016; Stiglitz, 2015).

Thinking about such data over the long term has appeared difficult, if not impossible, given that an appropriate and consistent long term housing & macro model has been lacking. The present paper aims to fill this void. We propose a *Long Term Housing & Macro Model* that enables us to analyze the mechanisms that produce the stylized facts on long term trends in house prices, land prices, the housing wealth-to-income ratio, and the split of private wealth into housing wealth and non-residential wealth.[2] We place considerable emphasis on modelling the housing sector and land as an input factor. Our new framework rests on three premises:

Premise 1 (Fixed Land Endowment). The overall land endowment is given by nature. The total amount of land that can be used economically, therefore, is fixed in the long run.

Premise 2 (Land Rivalry). Land that is used as an input in the production of new houses is permanently withdrawn from alternative economic uses, unless existing houses are demolished.

Premise 3 (Land in Housing Production). (i) A house is a bundle of the underlying land plot and the residential structure. Setting up new housing projects requires land as an essential input. (ii) Investments in structures do not, however, require land as an input.

[1] For a recent survey, focusing on business cycle and financial markets characteristics, see Piazzesi and Schneider (2016).

[2] We address stylized facts for four major advanced countries in Section 4. The ratio of housing wealth to total private domestic wealth in 2010 was 41.9% in the US, 63.3% in Germany, 63.2% in France, and 55.6% in the UK (Piketty and Zucman, 2014a).

Premise 1 represents a general law of nature. This has been acknowledged by previous researchers who model land and long run economic growth, starting with Nichols (1970).[3] A sceptic may argue that land augmenting technical change, land reclamation, and land development due to infrastructure investment can all enlarge the available amount of land. This is indeed plausible in the short to medium term. In the long run, however, the total amount of economically usable land is certainly fixed by nature. Increasing land scarcity is presently (and has been for decades) compatible with rising prices of farmland and urban residential land, as documented by Knoll et al. (2016) for 14 advanced countries, since World War II. This view is also compatible with the findings of Saiz (2010) who shows, by employing geographical, satellite generated data for U.S. metropolitan cities, that residential development is effectively curtailed by the availability of suitable land.[4] *Premise 2* simply describes the fact that land represents a rivalrous input: a plot of land that is underneath a house cannot, at the same time, host a manufacturing plant. As a result, land that is employed for setting up new housing projects resembles an exhaustible resource. *Premise 3 (i)* appears largely undisputed and is taken into account by existing theories (e.g., Davis and Heathcote, 2005; Favilukis, Sydney and Van Nieuwerburgh, 2016) that are discussed in Section 5 under the label *Canonical Housing & Macro Model. Premise 3 (ii)* has not yet been taken into account by existing models that treat land as an input for residential investment. Violating *Premise 3 (ii)* implicitly assumes that even replacement investments in depreciated structures (e.g., broken windows) require land. It also implies, however, a long run inconsistency in the sense that the cumulated amount of land devoted to the housing sector converges to infinity for time approaching infinity. In turn, this violates Premise 1 as well. We overcome this shortcoming by distinguishing between the extensive and the intensive margin of residential investment. Only an enlargement of the housing stock along the extensive margin, i.e., setting up new housing projects, requires land.

Our framework can be outlined as follows. We consider a two-sector Ramsey growth

[3] The popular statement "Buy land, they are not making it anymore", usually ascribed to Mark Twain, nicely illustrates this point.

[4] Zoning regulations and other restrictions on land use have inhibited the utilization of additional land in recent decades (Glaeser and Gyourko, 2003; Glaeser, Gyourko, and Saks, 2005). Moreover, Saiz (2010) stresses the positive interaction between (exogenous) land scarcity and (endogenous) regulations, implying that geographically constrained areas tend to be also highly regulated.

model. There is a numeraire goods sector and a housing sector. The paper's innovative contribution is the elaborate modeling of the housing sector. The housing stock can be expanded along the extensive margin (new housing projects), and along the intensive margin (larger residential buildings). Real estate development firms purchase non-residential land and transform it into residential land. There is land rivalry in the sense that land devoted to real estate development is permanently withdrawn from its alternative economic use, i.e. as an input in the numeraire goods sector. Housing services firms set up housing projects (extensive margin) by purchasing a developed real estate and combine it with residential buildings to produce housing services that are sold to households. The production of housing services, at the level of a single housing project, is characterized by decreasing returns to scale, as explained below. This creates profits for housing services producers, which provide incentives for real estate development in the first place, despite perfect competition. Construction firms produce residential structures (intensive margin) that represent an accumulable factor. The land price and the house price are fully endogenous and respond to economic growth that is driven by rising population density and technical change.

To illustrate the model's capabilities, we show that the calibrated model replicates, with remarkable accuracy, the historical evolution of housing wealth (relative to income) after Word War II. We also examine its implications for the future trajectory of housing wealth. Moreover, the model provides insights into the dynamics of non-residential wealth (relative to income), the second major private wealth component, consisting of physical capital and non-residential land wealth. The evolution of the wealth-to-income ratio and its decomposition is important for at least two reasons. *First*, Piketty (2014) stresses that a rising wealth-to-income ratio, assuming that the interest rate remains largely constant, changes the functional income distribution to the advantage of capital income recipients. Moreover, given that wealth is not uniformly distributed across the population, a rising wealth-to-income ratio is associated with a more unequal distribution of personal income.[5]

[5] Rognlie (2015) demonstrates that the increase in the economy-wide capital income share is driven exclusively by the housing sector. La Cava (2016) shows, employing data from Saiz (2010), that the long run increase in the aggregate share of housing capital income is driven critically by geographical and regulatory constraints on home building in larger US cities. Stiglitz (2015) points to the important role of land prices in the process of rising wealth-to-income ratios and increasing inequality of wealth and income.

Second, the rising wealth-to-income ratio appears to be a key trigger for the surge in the size of the finance industry, as documented by Philippon (2015). Jordá et al. (2016) argue that the growth of finance has been closely linked to an explosion of mortgage lending to households in the last quarter of the twentieth century. By analyzing the long run evolution of housing wealth, the paper therefore also provides a deeper understanding of the real and fundamental driving forces behind the process of financialization.[6]

Instead of providing an exhaustive overview of the theoretical literature on housing and macroeconomics, we highlight some of the more recent contributions. Hornstein (2008, 2009) employs a general equilibrium model to explain the surge in house prices in the US between 1975 and 2005. Davis and Heathcote (2005) build a multi-sector stochastic growth model with a housing sector to examine the business cycle dynamics of residential investment. Favilukis et al. (2016) construct a stochastic two-sector general equilibrium model of housing and non-housing production to explain the surge and the subsequent decline of the price-to-rent ratio in the US housing market between 2000 and 2010. Li and Zeng (2010) employ a two-sector neoclassical growth model with housing in order to explain a rising real house price driven by comparably slow technical progress in the construction sector. Borri and Reichlin (2016) use a two-sector, life-cycle economy with bequests to explain the increasing wealth-to-income ratio and wealth inequality driven by rising construction costs that result from comparably slow technological progress in the construction sector. We depart from the previous literature in two important respects that are related to each other. *First*, none of the previous contributions rests on the assumption that Premises 1-3 hold simultaneously. *Second*, we distinguish between the extensive and the intensive margin of housing production. As a result, we are able to provide a consistent and appropriate model that can be employed to analyze the long term.

The Canonical Housing & Macro Model, described by the previously mentioned papers, is well suited for analyzing phenomena at the business cycle frequency. However, it appears less suited to think about long term macroeconomic developments. The main reasons are that it does not capture rivalry between residential and non-residential land

[6] Gennaioli et al. (2014) show that the growth of finance can in fact be explained by a rising wealth-to-income ratio.

use (violating Premise 2) and leads to the long run inconsistency (by violating Premise 1 and Premise 3 (ii)), as mentioned above. Consequently, there are many important and intrinsically long term research questions that could not be addressed so far, such as: (1) How does a secular increase in housing demand affect the distribution of income and wealth? (2) How do zoning regulations (targeting the extensive margin of the housing stock) and building restrictions (targeting the intensive margin of the housing stock) affect housing affordability in a growing economy? (3) How does the taxation of housing wealth affect the dynamics of distribution of income and wealth? The Long Term Housing & Macro Model enables us to think about such important questions and we expect that our model will be employed in the future to address these and related research questions.

The paper is organized as follows. Section 2 presents our model. Section 3 derives important implications and defines the general equilibrium. Section 4 demonstrates how the model can be applied in order to understand the long term evolution of housing wealth, land prices, and house prices. It also discusses properties of the long run equilibrium.[7] Section 5 compares the Long Term Housing & Macro Model to the Canonical Housing & Macro Model. The final section concludes.

2 The Model

Consider a perfectly competitive, closed economy in general equilibrium. Time is continuous and indexed by $t \geq 0$. The innovation of our macroeconomic framework is the careful modeling of the housing sector. This sector encompasses different types of firms that interact in the production of housing services. *Real estate development firms* purchase a piece of land and conduct infrastructure investment to develop land for residential purposes. Real estate development provides the technical and legal prerequisite to produce housing services. It diminishes the amount of land that can be employed elsewhere in the economy. The overall supply of land, Z, is exogenously fixed. *Housing services firms* set up new housing projects by purchasing a developed real estate.[8] They produce housing

[7] Throughout the paper, we use "long run" for steady state values (as time goes to infinity), whereas "long term" refers to a time horizon that extends over several decades or even several centuries.

[8] Setting up one housing project requires to purchase one developed real estate.

services by combining a developed real estate and residential buildings ("structures"). *Construction firms* manufacture structures by employing materials and labor.

The numeraire sector produces a final good by combining physical capital, labor, and land. Like in standard (one-sector) models, the numeraire good can be used for (non-residential) consumption or for physical capital investments. It can also be transformed into materials that serve as input for building structures.

2.1 Firms

2.1.1 The Numeraire Good Sector

The *non-residential* (Y) *sector* produces a final good, chosen as numeraire, according to a standard Cobb-Douglas production function:

$$Y_t = B_t^Y (K_t^Y)^\alpha \left(L_t^Y\right)^\beta (Z_t^Y)^{1-\alpha-\beta}, \tag{1}$$

where K_t^Y, L_t^Y and Z_t^Y denote the amounts of physical capital, labor and land employed as input in the Y sector at time t, respectively. Total factor productivity (TFP) in the Y sector, $B_t^Y > 0$, may change over time and $\alpha, \beta > 0$ denote constant technology parameters that satisfy $\alpha + \beta < 1$. Physical capital is broadly defined to include non-residential structures, in addition to machines, and is only employed in the Y sector. The capital resource constraint is given by $K_t^Y \leq K_t$, where K_t denotes the total supply of physical capital in terms of the numeraire good. In equilibrium $K_t^Y = K_t$ will turn out to hold. Given that capital depreciates at rate $\delta^K \geq 0$, gross physical capital investment read as $I_t^K \equiv \dot{K}_t + \delta^K K_t$.[9] K_0 is given.

2.1.2 Housing Sector and the Characteristics of Land

There is free entry into the housing sector. Producing housing services requires to combine activities along the extensive margin (i.e., real estate development that constitutes an *ex ante* investment in the stock of houses) and along the intensive margin (i.e., producing residential buildings that may depreciate over time). Enlarging the stock of houses along

[9] A dot above a variable denotes the partial derivative with respect to time.

the extensive margin absorbs additional parcels of land that are incorporated into the newly built houses. Real estate development, therefore, inevitably implies that land is withdrawn from the alternative use in the Y sector.

Real Estate Development (Extensive Margin) The "number" of housing projects (houses) at time t is denoted by N_t, a real number. This variable captures the extensive margin of the housing stock. To set up one additional housing project (an activity that is conducted by housing services firms) it requires one developed real estate. We use the term "real estate development" (conducted by real estate development firms) to describe the activity of purchasing (non-residential) land and incurring (private) infrastructure investment that transform non-residential land into residential land. The number of land units that must be put underneath each house is given by $\psi > 0$. Total land usage in the housing sector is given by $Z_t^N = \psi N_t$. The land resource constraint reads as $Z_t^N + Z_t^Y \leq Z$, implying that the alternative use of land in the Y sector is limited by $Z_t^Y \leq Z - \psi N_t$. Consistent with *Premise 2 - land rivalry*, only land that has not been used in the process of real estate development can be devoted to the alternative land use (land area Z^Y), such as office space and land devoted to goods production, including agriculture, manufacturing (except construction), and services (except housing).

Let P_t^Z denote the price per unit of land. The costs $\mathcal{C}(\tilde{I}_t^N, P_t^Z)$ of increasing the number of developed real estates (equal to the number of housing projects) in any period t by

$$\dot{N}_t = \tilde{I}_t^N \tag{2}$$

amounts to

$$\mathcal{C}(\tilde{I}_t^N, P_t^Z) = \psi P_t^Z \tilde{I}_t^N + \frac{\xi}{2}\left(\tilde{I}_t^N\right)^2, \tag{3}$$

$\xi > 0$. The cost function (3) has two components. The first component, $\psi P_t^Z \tilde{I}_t^N$, shows the costs associated with the purchase of $\psi \dot{N}_t$ units of land. The second component, $\frac{\xi}{2}(\tilde{I}_t^N)^2 \equiv I_t^N$, gives the transformation costs (or adjustment costs) that equals the private infrastructure investment in terms of the numeraire good to transform non-residential land into residential land. These adjustment costs are convex in the number of newly developed real estates per period of time, $\dot{N}_t = \tilde{I}_t^N$, which makes N_t a state variable.

If we assumed that $\xi = 0$, then N_t would turn into a jump variable, which appears economically less plausible. N_0 is given.

As will become apparent, the land requirement per house (measured by parameter ψ) does neither affect the long run land price nor important long run ratios, such as the price-to-rent ratio and the different wealth-to-income ratios.[10] The distinction between the enlargement of the housing stock along the extensive margin (which requires a fixed amount of land) and the intensive margin (which does not require land) is consistent with *Premise 3 - land in housing production.* It allows us to treat the total available amount of land as fixed (*Premise 1 - land endowment*) despite continuous depreciation of the housing stock (along the intensive margin). This feature provides a key advantage vis-à-vis the Canonical Housing & Macro Model that is important when it comes to long term analysis, as explained in detail in Section 5.

Construction and Housing Services (Intensive Margin) Producing housing services requires to purchase a developed real estate (i.e., setting up a housing project) and combine it with residential buildings ("structures"). At the microeconomic level, the developed real estate represents the fixed input, whereas structures represent the variable input in the production of housing services. The amount of housing services produced increases with the amount of residential structures employed. However, reflecting the developed real estate as fixed input, it increases less than proportionate with the amount of residential structure.[11] That is, the production of housing services, at the level of single housing project, is characterized by decreasing returns to scale. Let x_t denote the amount of structures per housing project at time t. An amount x_t of structures produces housing services h_t per house according to

$$h_t = B_t^h (x_t)^\gamma, \tag{4}$$

[10]It does also not affect the labor share in total income, the economy's savings rate, and the factor allocation in the long run. See Appendix 7.2.

[11]Consider two houses (single-story versus two-story), each located on a piece of land of the same size. The two-story house produces a higher amount of housing services per period of time, but it produces less than twice the amount of the single-story house. For instance, living in the second floor may be viewed less convenient because both tenants and consumption goods must climb the height in everyday life. Moreover, the available parking space and / or garden space per square meter of living space declines.

$0 < \gamma < 1$, where $B_t^h > 0$ is a (possibly time-variant) productivity parameter. Total supply of housing services is $N_t h_t$.[12] There are two admissible institutional settings describing the relationship between housing services producers and real estate developers. First, both activities are organized within the same firm. Second, housing services producers buy real estates from real estate developers at price q_t^N that, in equilibrium, equals the present discounted value (PDV) of future profits accruing from a housing project.

There is a representative construction firm producing structures that are rented out to the housing services producers. It combines materials M_t and labor L_t^X according to a constant-returns-to-scale technology. The production of one unit of construction materials (e.g., cement) requires one unit of the numeraire good at time t.[13] That is, the extraction of construction material is implicitly assumed to require capital, labor, and land with the same technology as in the numeraire good sector. Let $\delta^X > 0$ denote the depreciation rate of structures (residential buildings) and

$$I_t^X = B_t^X (M_t)^\eta \left(L_t^X\right)^{1-\eta} \tag{5}$$

gross investment in structures, $0 < \eta < 1$, where $B_t^X > 0$ is a (possibly time-variant) productivity parameter. The total stock of residential structures, denoted by X_t, therefore evolves according to

$$\dot{X}_t = \underbrace{B_t^X (M_t)^\eta \left(L_t^X\right)^{1-\eta}}_{I_t^X} - \delta^X X_t \tag{6}$$

with X_0 given. The amount of residential buildings that is employed by all housing services firms cannot exceed the overall stock of residential buildings, i.e., $N_t x_t \leq X_t$.

Let q_t^X denote the shadow price per unit of structures associated with constraint (6). The house price is conceptualized as the sum of the value of a housing project (q_t^N) and

[12] We abstract from heterogeneity among housing projects. This feature simplifies the analysis as we do not have to keep track of the history of houses. This simplifying assumption is appropriate, given that we are not interested in the size distribution of firms in the housing sector.

[13] As we show in the working paper version (Grossmann and Steger, 2016), the marginal rate of transformation between materials and the numeraire good does not enter any outcome variable of interest in the long run and is thus implicitly calibrated to unity.

the value of the employed structure associated with a real estate (valued at $q_t^X x_t$), i.e.,[14]

$$P_t^H \equiv q_t^N + q_t^X x_t. \tag{7}$$

2.1.3 Households

There is an infinitely living, representative household with intertemporal utility

$$U = \int_0^\infty (\log C_t + \theta \log S_t)\, e^{-\rho t} \mathrm{d}t, \tag{8}$$

where C_t and S_t denote total consumption of the numeraire good and housing services at time t, respectively, $\rho > 0$ is the subjective discount rate, and $\theta > 0$ indicates the relative preference for housing services.[15] In equilibrium, $S_t = N_t h_t$.

Households supply inelastically L_t units of labor at time t to a perfect labor market. The labor resource constraint is $L_t^X + L_t^Y \leq L_t$. We allow L_t to increase temporarily and assume that it remains constant in the long run.

Households own the entire stock of financial assets, consisting of bonds that provide firms in the numeraire sector with physical capital (K_t), shares issued by housing services firms ($q_t^N N_t$), and ownership claims on construction firms ($q_t^X X_t$). The total financial asset holding, A_t, of the representative individual is thus given by

$$A_t = K_t + q_t^N N_t + q_t^X X_t. \tag{9}$$

The household supplies inelastically the non-residential land, i.e., the amount of land not (yet) purchased by the housing sector, to the numeraire good sector. Hence, in equilibrium, it holds that $Z_t^Y = Z - \psi N_t > 0$. Total private wealth, W_t, is the sum of financial asset holdings, A_t, and the value of non-residential land $P_t^Z Z_t^Y$. Thus, according

[14] The house price P_t^H can also be viewed as an ideal price index that is implicitly defined by $P_t^H N_t = q_t^N N_t + q_t^X X_t$.

[15] The instantaneous utility function is a (monotonic transformation of a) linearly homogenous function, preferences are homothetic. There exists a (positive and normative) representative consumer who owns the aggregate wealth and articulates the aggregate demand functions (e.g. Mas-Colell, Whinston and Green 1995, Chapter 4). Consequently, the individual distribution of assets and land does not play a role for the evolution of aggregates.

to (9) and (7), total private wealth may be expressed as

$$W_t \equiv A_t + P_t^Z Z_t^Y = \underbrace{P_t^H N_t}_{\text{housing wealth}} + \underbrace{K_t + P_t^Z Z_t^Y}_{\text{non-residential wealth}}, \tag{10}$$

where $P_t^H N_t$ (the house price times the number of houses) represents housing capital and $K_t + P_t^Z Z_t^Y$ captures non-residential wealth as the sum of physical capital and non-residential land wealth. Although initial values of stocks K_0, N_0, X_0 and Z_0^Y are given, total initial wealth, $W_0 = q_0^N N_0 + q_0^X X_0 + K_0 + P_0^Z Z_0^Y$, is endogenous because initial asset prices q_0^N, q_0^X, and the initial land price, P_0^Z, are endogenous.

To enable a careful model calibration, it is important to account for capital income taxation. The reason is that a tax on capital income affects the rate at which the profit stream of firms and land returns are discounted. We assume that both capital income and returns from land ownership are taxed at the constant rate τ_r.[16] For simplicity, we do not model government consumption or public investment and assume that the tax revenue is redistributed lump sum to households.

Let w_t, r_t, R_t^Z, p_t, and T_t denote the wage rate, the interest rate, the rental rate of land, the (relative) price per unit of housing services, and the lump sum transfer at time t, respectively. The intertemporal household budget constraint may then be expressed as

$$\dot{W}_t = (1 - \tau_r) r_t W_t + w_t L_t - C_t - p_t S_t + T_t. \tag{11}$$

2.1.4 Sources of Economic Growth

Sectoral productivities, as described by the vector $\mathbf{B}_t \equiv (B_t^Y, B_t^h, B_t^X)$, and population size, L_t, may (exogenously) grow for an arbitrary long period of time but approach a constant as $t \rightarrow \infty$. These processes are anticipated by all agents. We think it is appropriate to assume that population size cannot grow without bounds in an economy with fixed land endowment. The numerical analysis below (Section 4) is based on the assumption that B^Y, B^X and L increase for more than a century according to a logistic

[16] Labor income taxation would not have incentive effects and would not enter the reduced-form dynamic system (summarized in Online-Appendix A.2).

function that is chosen as part of our calibration strategy (keeping B^h time invariant).[17] One could, alternatively, allow for permanent exponential growth of B^Y and B^X. In this case, a steady state still exists, given that a specific parameter condition holds, as is typical for multi-sector growth models.

3 Equilibrium Analysis

This section highlights important equilibrium implications that result from the decisions of firms in the housing sector and defines the general equilibrium.

3.1 Decisions in the Housing Sector and Asset Prices

Let the rental rate of residential structures at time t be denoted by R_t^X. The instantaneous profit resulting from a housing project that accrues to *housing services producers*, noting production function (4), depends on the amount of employed structures and is given by $\pi_t \equiv p_t B_t^h (x_t)^\gamma - R_t^X x_t$. The necessary first order condition for profit maximization yields the inverse demand schedule for structures which is articulated by the representative housing services firm

$$R_t^X = p_t B_t^h \gamma (x_t)^{\gamma-1}, \tag{12}$$

implying positive equilibrium profits that amount to

$$\pi_t = (1-\gamma) p_t B_t^h (x_t)^\gamma. \tag{13}$$

The *representative real estate development firm* maximizes, at each time t, profits, defined as the value of newly developed real estates, $q_t^N \dot{N}_t$, minus total real estate development costs, $\mathcal{C}$.[18] Thus, using (2), the representative firm solves

$$\max_{\tilde{I}_t^N} q_t^N \tilde{I}_t^N - \mathcal{C}(\tilde{I}_t^N, P_t^Z) \quad \text{s.t.} \quad (3), \tag{14}$$

[17] Also if L and $\mathbf{B}$ were constant from the beginning, output would still grow as long as state variables X_t, K_t, N_t are below their long run levels (neoclassical convergence).

[18] Recall that it requires one developed real estate to set up one new housing project. Hence, the number and the value of housing projects are identical to the number and the value of developed real estates, respectively.

taking the price of developed real estates, q_t^N, and the land price, P_t^Z, as given. The associated first order condition implies the following law of motion (with N_0 given) for the number of developed real estates:

$$\left[\tilde{I}_t^N =\right] \dot{N}_t = \frac{q_t^N - \psi P_t^Z}{\xi}. \tag{15}$$

According to (15), if the value of a housing project is sufficiently large relative to the land price (i.e., if $q_t^N / P_t^Z > \psi$), the number of houses is being enlarged, i.e., $\dot{N}_t > 0$. Because the market for developed real estates is competitive, the price q_t^N will be bid up until it is equal to the PDV of the profit stream that the representative housing services firm can realize.[19] At every date t, it must therefore be true that $q_t^N = \int_t^\infty \pi_s e^{\int_t^s -r_v \mathrm{d}v} \mathrm{d}s$. Hence, the following no-arbitrage condition must hold at each t:

$$\frac{\dot{q}_t^N}{q_t^N} + \frac{\pi_t}{q_t^N} = r_t. \tag{16}$$

It says that the sum of the share price growth rate (capital gains) and the dividend (per unit of numeraire good invested) paid to the owners of housing services firms must equal the rate of return to bonds.[20] Therefore, households are indifferent between investing in housing projects or purchasing bonds.

Construction firms rent out the entire stock of residential structures to housing services producers by charging R_t^X per unit of structures. The *representative construction firm* maximizes the PDV of the cash flow, defined as the difference between rental income $R^X X$ and the costs of gross investment, $M + wL^X$. That is, the representative construction firm solves

$$\max_{\{M_s, L_s^X\}_{s=t}^\infty} \int_t^\infty \left(R_s^X X_s - M_s - w_s L_s^X\right) e^{\int_t^s -r_v \mathrm{d}v} \mathrm{d}s \quad \text{s.t.} \quad (6),\, X_0 \geq 0 \text{ given.} \tag{17}$$

[19] Notice the following analogy between our Long Term Housing & Macro Model and the horizontal innovation model by Romer (1990). Housing services producers must, at first, purchase a developed real estate (i.e. undertake an *ex ante* investment), at price q_t^N, before they can start to produce and market housing services to generate a stream of profits $\{\pi_t\}_{t=0}^\infty$. In the Romer (1990) model intermediate goods firms must, at first, purchase a blueprint before they can produce intermediate goods to generate a sequence of profits.

[20] Since all kinds of capital income are taxed at the same rate, tax rate τ_r does not enter (16).

From (17), it is easy to see (shown in Appendix 7.3) that the equilibrium rate of return per unit of structures satisfies

$$\frac{\dot{q}_t^X}{q_t^X} + \frac{R_t^X}{q_t^X} - \delta^X = r_t, \tag{18}$$

where q_t^X denotes the value of one unit of X. Ruling out bubbles by imposing an appropriate endpoint condition, the value per unit of structures equals the PDV of future rental returns, accounting for the depreciation rate δ^X of structures, i.e.,

$$q_t^X = \int_t^{\infty} R_s^X e^{\int_t^s -(r_v+\delta^X)\mathrm{d}v} \mathrm{d}s. \tag{19}$$

Implied by the constant returns to scale technology (5) and perfectly competitive markets, the value of total gross output in the construction sector must equal the total factor costs in construction, i.e.,

$$q_t^X I_t^X = M_t + w_t L_t^X. \tag{20}$$

Finally, denoting by R_t^Z the rental rate of land, the price of land equals the PDV of income from renting out one unit of land to the producers in the Y sector, i.e., $P_t^Z = \int_t^{\infty} R_s^Z e^{\int_t^s -r_v \mathrm{d}v} \mathrm{d}s$. Hence, the following no-arbitrage condition must hold:

$$\frac{\dot{P}_t^Z}{P_t^Z} + \frac{R_t^Z}{P_t^Z} = r_t. \tag{21}$$

3.2 Definition of Equilibrium and GDP

Definition 1. *A general equilibrium is a sequence of quantities, a sequence of prices, and a sequence of operating profits of housing services producers*

$$\{Y_t, K_t^Y, X_t, N_t, x_t, h_t, M_t, L_t^Y, L_t^X, Z_t^Y, C_t, S_t, A_t, W_t\}_{t=0}^{\infty},$$
$$\{p_t, P_t^Z, q_t^N, q_t^X, w_t, r_t, R_t^Z, R_t^X\}_{t=0}^{\infty}, \ \{\pi_t\}_{t=0}^{\infty}$$

for initial conditions $K_0 > 0$, $N_0 > 0$, $X_0 > 0$ *and given* $\{L_t, B_t^Y, B_t^X, B_t^h\}_{t=0}^{\infty}$ *such that*

1. *the representative individual maximizes lifetime utility, i.e., solves*

$$\max_{\{C_t,S_t\}_{t=0}^{\infty}} \int_0^{\infty} (\log C_t + \theta \log S_t)\, e^{-\rho t}\mathrm{d}t \;\; \text{s.t. (11)},\; W_t \exp\left(-\int_0^t (1-\tau_r) r_s \mathrm{d}s\right) \geq 0; \tag{22}$$

2. *the representative firm in the numeraire goods (Y) sector maximizes profits taking factor prices as given (i.e., factor prices equal marginal products);*
3. *the representative real estate developer solves profit maximization problem (14), taking the land price, P_t^Z, and the price of housing projects, q_t^N, as given;*
4. *housing services producers maximize profits at each time t, taking the price of housing services, p_t, and the rental rate of structures, R_t^X, as given;*
5. *the representative firm in the construction (X) sector solves profit maximization problem (17), taking the sequences of rental rate of structures $\{R_t^X\}_{t=0}^{\infty}$ and wage rates $\{w_t\}_{t=0}^{\infty}$ as given;*
6. *there are no arbitrage possibilities from purchasing land, i.e., (21) holds;*
7. *the bond market, the land market, the market for structures, and the land market clear at any t, i.e.,*

$$K_t^Y = K_t, \tag{23}$$
$$Z_t^Y = Z - \psi N_t, \tag{24}$$
$$N_t x_t = X_t, \tag{25}$$
$$L_t^X + L_t^Y = L_t; \tag{26}$$

8. *the financial asset market clears at any t, i.e., (9) holds, and total wealth is given by (10);*
9. *the market for housing services clears at any t, i.e.,*

$$S_t = N_t h_t = N_t B^h (x_t)^{\gamma}; \tag{27}$$

10. *the market for the numeraire good clears at any t, i.e.,*[21]

$$Y_t = C_t + I_t^K + I_t^N + M_t. \tag{28}$$

The reduced form dynamic system that results from this equilibrium definition comprises seven differential equations plus a set of algebraic equations.[22]

[21] Equilibrium condition (28) is redundant, according to Walras' law. To exclude conceptual or calculation errors, we prove, in Online-Appendix A.1, that the long run equilibrium derived from conditions 1-9 fulfills condition 10.

[22] It is summarized in Online-Appendix A.2.

The gross domestic product (GDP) is given by the value of total consumption, $\mathcal{C}_t \equiv C_t + p_t S_t$, plus the value of total investment, $\mathcal{I}_t \equiv I_t^K + I_t^N + q_t^X I_t^X$. Using (20), (27) and (28) in $GDP_t = \mathcal{C}_t + \mathcal{I}_t$ gives us the GDP as the sum of value added of all sectors:

$$GDP_t = Y_t + p_t N_t h_t + w_t L_t^X. \tag{29}$$

4 Thinking Long Term

How can we employ the model to think about the, by now available, time series data on housing and macro over the long term? In the main text, we focus on a few selected variables that appear particularly interesting in light of recent discussions, namely the housing wealth-to-income ratio, the house price, and the land price.

Piketty and Zucman (2014b) document the long term evolution of the wealth-to-income ratio, while the long term evolutions of the aggregate real land price and the aggregate real house price have been investigated by Davis and Heathcote (2007) and Knoll et al. (2016). The long term surge in real asset prices and wealth (relative to income) after World War II is of macroeconomic importance as it has the clear potential to unfold first order distributional consequences. For instance, rising land prices point to a modern version of Ricardo's (1817) famous principle of scarcity.[23] While Ricardo was mainly concerned with agricultural land and the production of corn to feed a growing population, societies in modern times are to a larger extent confronted with the need for residential investments to meet the increasing demand for housing services under the constraint of scarce land.

Thinking long term requires two ingredients: *First*, a model that allows for the determination of a consistent steady state (Section 4.1). *Second*, a model that can also be employed to trace the evolution of the variables under study over time (Section 4.2).

[23] Ricardo (1817) argued that, over the long term, economic growth benefits landlords disproportionately, as the owners of the fixed factor. Since land is unequally distributed across the population, Ricardo reasoned that market economies would produce rising inequality (see also Piketty, 2014).

4.1 Long Run Equilibrium

According to Proposition 1 in Appendix 7.2, a long run equilibrium (steady state) typically exists, is unique and characterized by zero growth rates of all stock variables.[24] Here we highlight only a few selected steady state implications.

Define by $D \equiv L/Z$ the population density and by $D^Y \equiv L^Y/Z^Y$ the labor-to-land ratio in the Y sector. The long run land price, P^{Z*}, and the long run house price, P^{H*}, read as[25]

$$P^{Z*} = \frac{(1-\alpha-\beta)}{r^*}\left(\frac{\alpha}{r^*+\delta^K}\right)^{\frac{\alpha}{1-\alpha}}\left(B^Y\right)^{\frac{1}{1-\alpha}}\left(D^{Y*}\right)^{\frac{\beta}{1-\alpha}}, \tag{30}$$

$$P^{H*} = \frac{\psi(1-\alpha-\beta)\left(\frac{\alpha}{r^*+\delta^K}\right)^{\frac{\alpha}{1-\alpha}}\left(B^Y\right)^{\frac{1}{1-\alpha}}\left(D^{Y*}\right)^{\frac{\beta}{1-\alpha}}\left(\frac{1}{1-\gamma}+\frac{\delta^X}{r^*}\right)}{r^*+\delta^X}, \tag{31}$$

where the long run real interest rate is given by $r^* = \frac{\rho}{1-\tau_r}$ and the long run labor-to-land ratio in the Y sector, D^{Y*}, is proportional to D, but independent of TFP parameters, such as B^Y.[26] Thus, both P^{Z*} and P^{H*} depend positively on population density, mirroring Ricardo's (1817) principle of scarcity.

We follow Piketty (2014) and Piketty and Zucman (2014a, 2014b, 2015) and employ the net domestic product (NDP) as aggregate income in the denominator to report wealth-to-income ratios (Appendix 7.1). NDP equals GDP net of depreciation, $NDP = GDP - \delta^K K - \delta^X q^X X$. Housing wealth is given by $P^H N$. The long run housing wealth-to-income ratio, $\mathfrak{H}^{NDP} \equiv \frac{P^H N}{NDP}$, is given by:

$$\mathfrak{H}^{NDP*} = \frac{\left(1+\frac{(1-\gamma)\delta^X}{r^*}\right)\left[r^*+\delta^X+\left(\frac{r^*+\delta^X}{\theta}+\eta\gamma\delta^X\right)\frac{r^*+\delta^K}{r^*+(1-\alpha)\delta^K}+(1-\eta)\gamma\delta^X\right]}{\left[r^*+\left(\frac{r^*+\delta^X}{\theta}+\eta\gamma\delta^X\right)\frac{r^*+\delta^K}{r^*+(1-\alpha)\delta^K}+\left(1+(1-\eta)\gamma\right)\delta^X\right]\left(1+\frac{1}{\theta}\right)\left(r^*+\delta^X\right)}. \tag{32}$$

Importantly, $\mathfrak{H}^{NDP*}$ neither depends on population density, D, nor on TFP parameters. Appendix 7.2 (Steady State Properties) and Appendix 7.3 (Proofs) demonstrate that this

[24] Online-Appendix A.3 provides a complete analytical steady state characterization.

[25] Throughout, superscript (*) denotes long run equilibrium values of endogenous variables.

[26] In Appendix 7.3, we show that $D^{Y*} = D\frac{\frac{r^*+\delta^X}{\theta}+\eta\gamma\delta^X+\frac{r^*+(1-\alpha)\delta^K}{r^*+\delta^K}\frac{(1-\gamma)(r^*+\delta^X)}{1-\alpha-\beta}}{\frac{r^*+\delta^X}{\theta}+\eta\gamma\delta^X+\frac{r^*+(1-\alpha)\delta^K}{r^*+\delta^K}\frac{(1-\eta)\gamma\delta^X}{\beta}}$.

property applies, in the long run, to all interesting ratios of endogenous variables, such as the non-residential wealth-to-income ratio, $\mathfrak{N}^{NDP} \equiv \frac{K+P^Z Z^Y}{NDP}$, the NDP-to-GDP ratio, $\frac{NDP}{GDP}$, the labor share in GDP, $\frac{wL}{GDP}$, the investment rate, $\frac{\mathcal{I}}{GDP}$, and the ratio of total housing consumption to total housing wealth, $\frac{pS}{P^H N}$ (i.e., the inverse of the house price-to-rent ratio). The same holds for the long run allocation of labor and land, characterized by the fractions L^X/L, L^Y/L, Z^N/Z, and Z^Y/Z. These steady state properties imply that we can assess, say, the long run housing wealth-to-income ratio without knowing D (the ratio of the population size to the amount of economically usable land) or TFP levels in the long run.[27]

4.2 Transitional Dynamics - Post World War II

The calibrated Long Term Housing & Macro Model is solved numerically. Appendix 7.4 describes the calibration strategy and documents the data sources. We trace the historical evolution of the housing wealth-to-income ratio, the house price, and the land price after World War II and provide projections until the year 2100. The numerical analysis employs the relaxation algorithm to solve the model for transitional dynamics (Trimborn, Koch and Steger, 2008). This technique allows us to calculate exact numerical solutions for the transitional dynamics implied by the non-linear differential equation system.[28]

4.2.1 The Experiment

The macroeconomic, model-based experiment – conducted separately for France, Germany, UK and the USA – can be described as follows: *First*, we feed country-specific population growth, as reported by Piketty and Zucman (2014b), into the model.[29] We also feed in country-specific time paths for TFP parameters B_t^X and B_t^Y according to a

[27] Long run implications of the calibrated model for a variety of interesting variables are presented in Online-Appendix A.4.

[28] This procedure is extended to analyze sizeable transitions that are driven by large shocks in state variables and substantial exogenous movements in population size and TFP levels, which is appropriate for the period after World War II. The relaxation algorithm is implemented in Mathematica and the code is available upon request.

[29] For US, UK and France, we normalize land size to unity. We account for the German reunification by raising land size in the year 1990 by 40 percent along with the observed increase in population size.

logistic function, such that, given the observed (exogenous) increases in population size and endogenous capital accumulation, implied aggregate income growth rates between 1955 and 2010 coincide with the actual growth rates, reported by Piketty and Zucman (2014b).[30] That is, economic growth is driven by exogenous population growth, exogenous technical progress, and endogenous capital accumulation. Growth is transitory, but may extend over several centuries. *Second*, initial state variables are set to match initial wealth-to-income ratios. Specifically, N_0 and X_0 are set such that the model-based initial housing wealth-to-income ratio, $\mathfrak{H}_0^{NDP}$, matches the respective empirical value in 1955, assuming identical proportional deviations from the initial steady state (given L_0, B_0^Y and B_0^X) for both N_t and X_t. Similarly, K_0 is set such that model-based non-residential wealth-to-income ratio, $\mathfrak{N}_0^{NDP}$, matches the respective empirical value in 1955.[31] We let the simulation run from 1955 until 2100 to yield country-specific time paths for $\mathfrak{H}_t^{NDP}$ for that time period.

4.2.2 Housing Wealth-to-Income Ratio

Figure 1 displays the evolution of the housing wealth-to-NDP ratio, $\mathfrak{H}_t^{NDP}$, for France, Germany, UK, and the US, over time. The solid (blue) lines display the model-based time paths, whereas the dotted (red / purple) lines represent the empirical data.[32] Overall, the model matches the empirical series quite accurately, given that only information about initial state variables, population growth and aggregate income growth enters the experiment and the calibration of the model (described in Appendix 7.4) does not rely on housing wealth. There are, of course, deviations between the empirical data and the

[30] To capture the evidence that real construction costs have slightly risen over time, we assume that the increases over time in B^X amount to 80 percent of the growth in B^Y. As TFP parameter B^h concerns the transformation of structures per house into housing services, it does not help to match observables. We thus capture productivity improvements in the housing sector solely by increases in B^X and normalize $B^h = 1$.

[31] These initial state variables are specified as percentage of the initial long run equilibrium values, K_0^*, X_0^*, N_0^*, that would result for initial population size and productivity levels, L_0, B_0^Y, B_0^X. Starting out of steady state appears especially plausible for European economies shortly after World War II.

[32] Empirical data are displayed as decadal averages to smooth out business cycle fluctuations. The last five entries (in purple color) represent interpolations between the 2005 decadal average and the actual value in 2010.

results of the model. The strongest deviation can be observed for Germany that starts in 1955 with $\mathfrak{H}_{1955}^{NDP} \approx 0.65$ and in recent times shows $\mathfrak{H}_{2005}^{NDP} \approx 2.19$, whereas the model implies $\mathfrak{H}_{2005}^{NDP} \approx 3.33$. Given $\mathfrak{H}_{1955}^{NDP} \approx 0.65$, Germany starts with the lowest value of the housing stock, which appears reasonable with respect to war destructions during World War II.[33] The model economy then builds up the housing stock quite rapidly, despite convex (quadratic) land transformation costs. The model-based values nearly match the empirical observations in 2005 for FRA and UK. For the US, the empirical value in 2005 is about $\mathfrak{H}_{2005}^{NDP} \approx 2.18$, while the model implies $\mathfrak{H}_{2005}^{NDP} \approx 2.50$, a deviation of about 15 percent. The long run value is the same for every economy, given that we assume a unique capital income tax rate of $\tau_r = 0.2$, and amounts to about 410 percent. The model therefore gives us, for the first time, a notion about the long run housing wealth-to-income ratio and the specific trajectory that converges to this asymptotic value.[34]

In a similar fashion, one can investigate the evolution of non-residential wealth (relative to income) over time, $\mathfrak{N}_t^{NDP}$.[35] The Long Term Housing & Macro Model therefore enables us to analyze the major private wealth-to-income ratios, as discussed by Piketty (2014) and Piketty and Zucman (2014a, 2015), within a unifying theoretical model of different wealth components.

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[33] Piketty and Zucman (2014b) report war destructions of about 50 percent for the housing stock and about 27 percent for physical capital.

[34] Recall that population growth and / or TFP growth affects the transition to the steady state values, but not the steady state values itself. Online-Appendix A.5 demonstrates that also the time path of the housing wealth-to-income ratio changes very little if we assume that the amount of economically usable land changes over time.

[35] To save space, the analysis is relegated to Online-Appendix A.6. In addition, Online-Appendix A.4 reports the split of non-residential wealth between physical capital, K, and non-residential land wealth, $P^Z Z^Y$, in the long run.

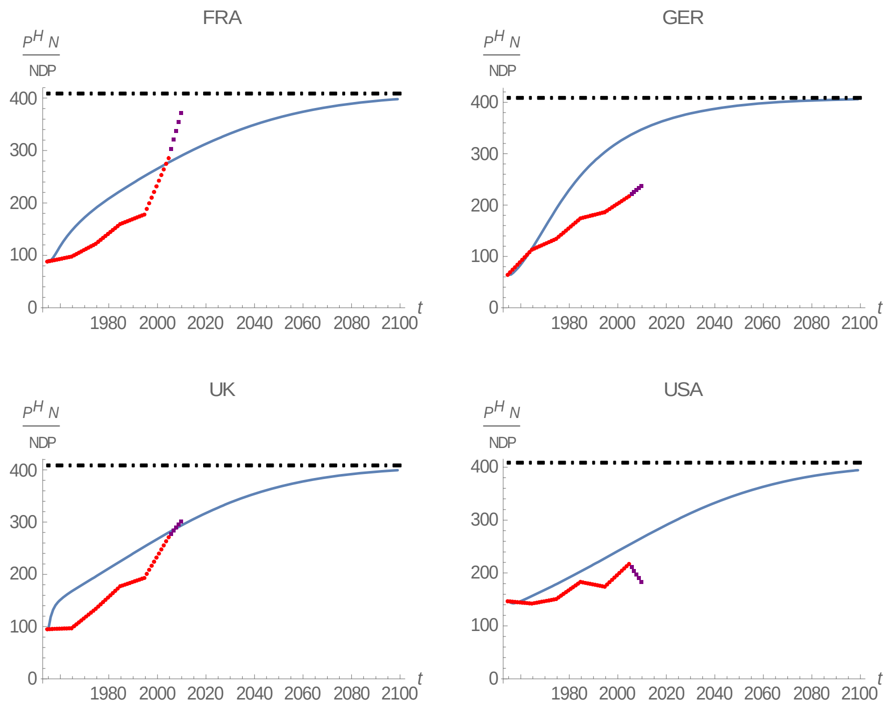

Figure 1. Housing wealth (relative to NDP) from 1955 until 2100.

Notes. (1) Dotted line: empirical series (red: linear interpolation between decadal averages starting in 1955 until 2005; purple: linear interpolation between the 2005 value and the actual 2010 value). Solid (blue) line: implied series resulting from the calibrated Long Term Housing & Macro Model. (2) Country specific parameters: N_0 and X_0 are set such that model-based and empirical values for $\frac{P_0^H N_0}{NDP_0}$ coincide, assuming the same proportional deviations from the initial steady state. K_0 is set such that model-based and the empirical values for $\frac{K_0+P_0^Z Z_0^Y}{NDP_0}$ coincide. Population grows according to a logistic function in line with empirical population growth. TFP parameters B_t^Y and B_t^X increase over time according to logistic functions such that GDP grows in accordance with empirical data between 1955 and 2010, given (exogenous) population growth and (endogenous) capital accumulation. Data are taken from Piketty and Zucman (2014b). Land size is normalized to $Z = 1$ except for Germany where $Z = 1.4$ after the reunification in 1990. (3) Common parameters: $\alpha = 0.28$, $\beta = 0.69$, $\gamma = 0.9$, $\eta = 0.38$, $\theta = 0.22$, $\delta^X = 0.015$, $\delta^K = 0.07$, $\rho = 0.025$, $\tau_r = 0.2$, $\psi = 1$, $\xi = 100$.

4.2.3 Land Prices and House Prices

The model-based explanation for the evolution of housing wealth (relative to income) provided above has an important correlate, namely the evolution of land prices and house prices. The factor land plays a prominent role. Given that the overall land endowment is fixed (Premise 1) and that land represents a rivalrous factor (Premise 2), land is becoming scarcer and more expensive as the economy grows. Our analysis thus suggests that the price channel of increased land valuations plays a critical role in the process of pushing wealth-to-NDP ratios up over time. Rising land prices push the house price up and this price channel triggers an increase in the ratio of housing wealth to NDP, $\mathfrak{H}_t^{NDP}$.[36]

Figure 2 displays the evolution of the land price, P_t^Z, and the house price, P_t^H, as implied by the calibrated Long Term Housing & Macro Model, assuming that the labor force grows, in total, by a factor of 1.72 and TFP parameters B_t^Y and B_t^X rise by a factor of 2.7 and 2.16 between 1955 and 2010, respectively. These are average values across France, UK, and the US.[37] The model yields an overall increase in the land price by a factor of 7.5 and an overall increase in the house price by a factor of 3.4 between 1955 and 2010. This pattern is largely consistent with the empirical data. Considering averages across France, UK and the USA yields a growth factor of 5.2 for land prices and a growth factor of 3.2 for house prices between 1955 and 2010 (Knoll et al., 2016). Hence, the calibrated model is consistent with an additional important stylized fact, namely that the land price increases by more than the house price and therefore explains the major share of the surge in house prices.[38]

[36] In addition, the ratio of non-residential wealth to NDP, $\mathfrak{N}_t^{NDP}$, rises in the process of economic growth, because it contains a sizable land wealth component, $P_t^Z Z_t^Y$. See Online-Appendix A.6.

[37] We compare the implications of the calibrated model in Figure 2 with the imputed land prices provided in Knoll et al. (2016). As the German house price index in part relies on residential land prices, implying that imputed land prices cannot be calculated, Germany is excluded.

[38] Davis and Heathcote (2007) show that the price of residential land has increased considerably more than house prices during 1975-2005 in the US (almost fourfold), whereas the costs of structures have increased only slightly. Focusing on 14 countries between 1950 and 2012, Knoll et al. (2016) demonstrate that 80 percent of the increase in house prices can be attributed to rising land prices and only 20 percent to rising construction costs.

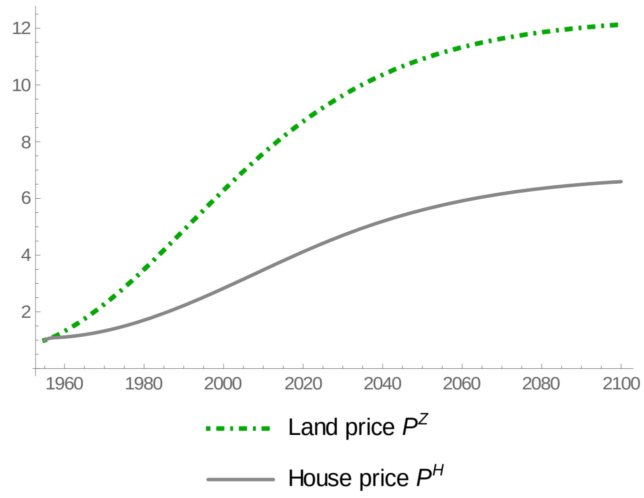

Figure 2. Land prices and house prices ("average economy"): Calibrated Long Term Housing & Macro Model.

Notes: Evolution of the land price, P_t^Z, and the house price, P_t^H, as resulting from the model (1955 values = 1). L_t increases, in total, by a factor of 1.72, while B_t^Y and B_t^X increase by a factor of 2.7 and 2.16, respectively, according to logistic functions. State variables (K_t, N_t and X_t) start below their (conditional) steady state, the proportional deviations are averaged across FRA, UK, and the US, according to Figure 1. Other parameters as for Figure 1.

5 Canonical Housing & Macro Model

We finally sketch and discuss the canonical macroeconomic model with a housing sector, as it has been recently employed, among others, in Davis and Heathcote (2005), Hornstein (2009), Iacoviello and Neri (2010), Li and Zeng (2010), Favilukis et al. (2016), and Borri and Reichlin (2016). The goal is to highlight the major differences between the Canonical Housing & Macro Model and our Long Term Housing & Macro Model.

5.1 Setup

The economy is perfectly competitive and comprises two sectors. The *numeraire (Y) sector* combines capital, K_t^Y, and labor, L_t^Y to produce a final output good according to

$$Y_t = B_t^Y \left(K_t^Y\right)^{\alpha} \left(L_t^Y\right)^{1-\alpha}, \tag{33}$$

$B_t^Y > 0$, $0 < \alpha < 1$. Notice that land does not enter the Y technology. The numeraire good can be either consumed or invested. The *housing sector* produces housing services that are sold to households. Housing services per period of time, S_t, are proportional to the stock of houses H_t, i.e., $S_t = bH_t$ with $b > 0$. Without loss of generality we set $b = 1$. The production of houses employs a fixed amount of (additional) land together with a variable amount of structures. The fixed amount of residential land becomes exogenously available each period. The stock of houses accumulates according to $\dot{H}_t = I_t^H - \delta^H H_t$, where I_t^H denotes gross investment and $\delta^H > 0$ the depreciation rate of the housing stock. Gross additions to the housing stock are described by a constant-returns to scale technology,

$$I_t^H = B_t^H X_t^{\beta} \bar{Z}^{1-\beta}, \tag{34}$$

$B^H > 0$, $0 < \beta < 1$, where X_t is the amount of structures that are combined with a fixed quantity of (additional) land, $\bar{Z}$, which is inelastically supplied each period. Residential structures (a flow) are produced according to

$$X_t = B_t^X \left(K_t^X\right)^{\gamma} \left(L_t^X\right)^{1-\gamma}, \tag{35}$$

$B^X > 0$, $0 < \gamma < 1$, by combining capital, K^X, and labor, L^X. Consequently, the evolution of the housing stock H is described by

$$\dot{H}_t = \underbrace{\tilde{B}_t^H \left(K_t^X\right)^{\beta\gamma} \left(L_t^X\right)^{\beta(1-\gamma)} \bar{Z}^{1-\beta}}_{=I_t^H} - \delta^H H, \tag{36}$$

where $\tilde{B}_t^H \equiv B_t^H \left(B_t^X\right)^{\beta}$. The exogenous and time-invariant supply of (additional) residential land, $\bar{Z}$, used as specific factor in housing production, is supposed to capture the idea that "a constant quantity [...] of new land/permits suitable for residential development is available each period" (Favilukis et al., 2016). Davis and Heathcote (2005) point to the "declining relative returns to agricultural use" (p. 756) as a potential source of new land.[39]

Households maximize utility and firms maximize profits. Intertemporal utility of the representative consumer is again given by (8). Let q_t^H denote the value per unit of H at time t. Total wealth is equal to the value of financial assets, comprising physical capital and housing wealth, i.e.,

$$W_t = A_t = K_t + q_t^H H_t, \tag{37}$$

where $K_0 > 0$ and $H_0 > 0$ are given. Again, let r_t denote the rate of return of financial assets, w_t the wage rate, R_t^Z the land rent, and p_t the price of housing services, respectively. Household wealth then accumulates according to

$$\dot{W}_t = r_t W_t + w_t L_t + R_t^Z \bar{Z} - p_t S_t - C_t. \tag{38}$$

The resource constraints (holding with equality in equilibrium) are given by

$$K_t^X + K_t^Y \leq K_t, \tag{39}$$

$$L_t^X + L_t^Y \leq L_t, \tag{40}$$

and market clearing in the numeraire good sector requires

$$Y_t = C_t + I_t^K = C_t + \dot{K}_t + \delta^K K_t. \tag{41}$$

GDP_t for this economy is the sum of value added of the numeraire good sector, Y_t, the

[39] The canonical model could alternatively be interpreted as a model with two consumption goods, one flow good and one durable (capturing something else than house capital) that is produced by making use of an intermediate product and a specific factor that could as well be viewed as specific labor (like in the Ricardo-Viner model).

housing services sector, $p_t H_t$, and the construction sector (building new houses with value $q_t^H I_t^H$).[40]

$$GDP_t = Y_t + p_t H_t + q_t^H I_t^H. \tag{42}$$

5.2 Comparison with the Long Term Housing & Macro Model

The Canonical Housing & Macro Model, due to its lean structure, represents an attractive and important analytical tool to study a large set of research questions in the business cycle context.[41] We argue, however, that this approach is less suited for the long term analysis. The Long Term Housing & Macro Model, on the other hand, is a bit more complex, but appropriate when it comes to research questions that focus on longer time horizons. We now highlight the major differences between the two models.

5.2.1 Land Availability and Land Allocation

Canonical Housing & Macro Model The quantity of (additional) land that is employed every period in the housing sector, $\bar{Z}$, is an exogenous and time invariant flow variable. Land is not used elsewhere in the economy. That is, there is no endogenous land allocation and non-residential land as a wealth component is not accounted for. This automatically implies that the amount of land allocated to the housing sector does not change along the transition. What may change along the transition to the steady state is the quantity of complementary factors: K_t^X and L_t^X.

Long Term Housing & Macro Model The economy-wide amount of land is fixed stock variable. Land can be either employed in the Y sector (Z_t^Y), or in the housing sector ($Z_t^N = \psi N_t$). The quantity of land allocated to the housing sector is time-varying and endogenous. This difference in the land allocation (exogenous and time-invariant vs.

[40] Alternatively, one can define GDP according to its use: $GDP_t = C_t + p_t S_t + I_t^K + q_t^H I_t^H$, which is implied by (42), according to (41) and $S_t = H_t$.

[41] In Online-Appendix B we summarize its dynamic system and derive analytical results for the long run equilibrium.

endogenous and time-varying) has important implications for the evolution of the land price, as explained below.

5.2.2 Cumulated Amount of Land Absorbed by a Stationary Housing Stock

Canonical Housing & Macro Model Let us assume a stationary steady state with a positive and constant housing stock and a positive depreciation rate $\delta^H > 0$. The cumulated amount of land that is incorporated in the housing sector converges to infinity as time goes to infinity. This is not compatible with Premise 1 (fixed overall land endowment) above and may be labelled long run inconsistency.

Long Term Housing & Macro Model Let us assume a stationary steady state with a positive and constant housing stock and a positive depreciation rate of residential buildings $\delta^X > 0$. The cumulated quantity of land that is incorporated in the housing stock is $Z_t^N = \psi N_t$, a finite number, even for time approaching infinity. Hence, the Long Term Housing & Macro Model is consistent with Premise 1 and the long run inconsistency is avoided.

5.2.3 Extensive and Intensive Margin of Housing Production

Canonical Housing & Macro Model The housing stock is a one-dimensional object. It appears appropriate to interpret an increase in the housing stock as an increase along the extensive margin because this process requires land. The alternative interpretation would imply that there is a single house that is enlarged continuously upwards.

Long Term Housing & Macro Model The housing stock can be enlarged along the extensive margin (increasing the number of houses) and along the intensive margin (increasing the size of the typical houses). Only the enlargement along the extensive margin requires land as an input (Premise 3).

The distinction between the extensive and intensive margin has three advantages: (i) It allows us to avoid the long run inconsistency of the canonical model that land input goes to infinity when there is depreciation of residential structures. (ii) It enables a distinction between a change of the housing stock either along the extensive margin (e.g., through real estate development or a natural disaster) or along the intensive margin (e.g., through a depreciated stock of structures or investment in structures). (iii) It enables us to distinguish the effects of regulations targeting the extensive margin of the housing stock (zoning regulations) and the intensive margin (building restrictions) within model-based policy evaluations.

5.2.4 Land Price Dynamics

To illustrate a major difference between the two models in the determination of the land price, which is important for our analysis of the previous section, we ask the following question: How does the land price evolve in response to a destruction in the housing stock?

Canonical Housing & Macro Model Given that land input is a (time-invariant) flow, the land price, P_t^Z, equals the competitive land rent, R_t^Z, i.e., $P_t^Z = R_t^Z = p_t \frac{\partial I_t^H}{\partial \bar{Z}}$ (Davis and Heathcote, 2005; Favilukis et al., 2016). Suppose that the initial housing stock is below the steady state level, $H_0 < H^*$. As a consequence, the economy allocates a large amount of capital, K_t^X, and labor, L_t^X, to the construction sector. This construction boom implies that the housing stock is built up. As the economy converges to the steady state, the construction boom diminishes, implying that capital and labor are being reallocated to the Y sector. Because the time-invariant flow of land, along the transition to the steady state, is combined with less and less K_t^X and L_t^X, the marginal productivity of land, $\frac{\partial I_t^H}{\partial \bar{Z}}$, declines.[42] Moreover, the price of housing services, p_t, declines too as the supply in the housing market is being enlarged. Taken together, the land price $P_t^Z = p_t \frac{\partial I_t^H}{\partial \bar{Z}}$ unambiguously decreases along the transition to the steady state, as illustrated in Figure 3 (a).

[42] Recall that gross residential investments may be expressed as: $I_t^H = \tilde{B}_t^H \left(K_t^X\right)^{\beta\gamma} \left(L_t^X\right)^{\beta(1-\gamma)} \bar{Z}^{1-\beta}$.

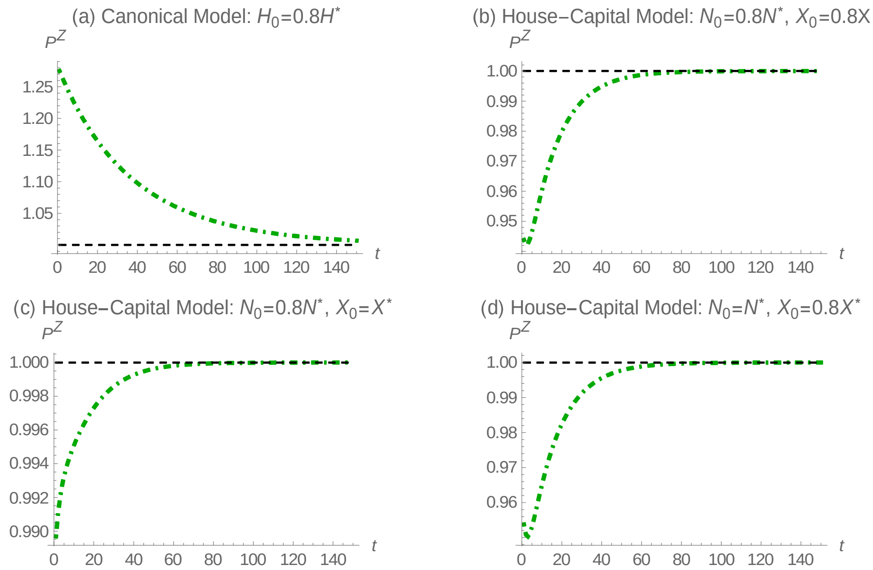

Figure 3. Evolution of the land prices in response to housing stock destruction: Canonical Housing & Macro Model and Long Term Housing & Macro Model.

Notes: Panel (a): Canonical Housing & Macro Model, initial housing stock $H_0 = 0.8H^*$, set of parameters: $\rho = 0.025$, $\gamma = 0.7$, $\delta^H = 0.015$, $\delta^K = 0.07$, $\alpha = 0.4$, $\beta = 0.5$, $\theta = 0.22$, $L = \bar{Z} = B^Y = B^X = 1$. Panel (b): Long Term Housing & Macro Model: $N_0 = 0.8N^*$ and $X_0 = 0.8X^*$; Panel (c): Long Term Housing & Macro Model: $N_0 = 0.8N^*$ and $X_0 = X^*$; Panel (d): Long Term Housing & Macro Model: $N_0 = N^*$ and $X_0 = 0.8X^*$; Panels (b) - (d): set of parameters other than initial states as in Figure 2 except $B^Y = B^X = L = 1$.

Long Term Housing & Macro Model Each unit of land can be either permanently incorporated in a house or can be employed for an infinite sequence of periods in the Y sector. The equilibrium land price, excluding bubbles, equals the PDV of an infinite land rent earned in the Y sector: $P_t^Z = \int_t^\infty R_\tau^Z e^{\int_t^s -r_v \mathrm{d}v} \mathrm{d}s$.

Figure 3 (b) shows that the land price *increases* along the transition to the steady state in response to a destruction of the housing stock in both dimensions, the intensive and the extensive margin. To understand why, first, suppose the number of houses N is initially below its steady state level ($N_0 < N^*$), given the amount of structures. Then there is initially more land available for the alternative use in the Y sector (Premise 2) than in the long run. A high Z_t^Y implies that its marginal product $R_t^Z = \frac{\partial Y}{\partial Z_t^Y}$ and thus land price P_t^Z are low early on. This triggers off a construction boom, implying that (i) labor is reallocated to the construction sector and (ii) physical capital is temporarily decumulated (a standard implication of multi-sector models with multiple state variables). That is, non-residential land, Z_t^Y, is initially combined with a small amount of complementary factors, K_t^Y and L_t^Y, again implying that P_t^Z is low. As more and more land is used, however, to build houses, Z_t^Y is diminished over time by real estate development (Premise 2). This leads to increasing land scarcity. Consequently, both $R_t^Z = \frac{\partial Y}{\partial Z_t^Y}$ and thus the land price P_t^Z increase over time, as shown in Figure 3 (c). (As the construction boom diminishes over time, also labor returns to the Y sector and the physical capital stock accumulates to its steady state level.) Both mechanisms cannot arise in the canonical model because there is neither an alternative use of land nor can land become scarcer over time. Second, suppose that the amount of structures X is initially lower than its steady state level ($X_0 < X^*$), for a given number of houses. In this case, due to decreasing returns in the production of housing services, the inverse demand for structures, R_0^X, is high initially. Again, this triggers off a construction boom that drives up the land price over time, as shown in Figure 3 (d), invoking similar mechanisms as for Figure 3 (c).

5.2.5 Housing Wealth and Non-Residential Wealth

Canonical Housing & Macro Model Housing wealth is given by $q_t^H H_t$. This wealth component comprises the cumulated quantity of land that is incorporated in the housing stock, similarly to the Long Term Housing & Macro Model (both residential land and structures included). The non-residential wealth-to-NDP ratio is given by $\mathfrak{N}_t^{NDP} = \mathfrak{K}_t^{NDP} = \frac{K_t}{NDP_t}$. That is, land outside the housing sector, i.e., the value of non-residential land, is missing (i.e., $P_t^Z Z_t^Y \equiv 0$).

Long Term Housing & Macro Model Residential land, $Z_t^N = \psi N_t$, enters housing wealth, $P_t^H N_t = q_t^N N_t + q_t^X X_t$, since land is incorporated in houses. Non-residential land, Z_t^Y, enters non-residential wealth, that in total is given by $K_t + P_t^Z Z_t^Y$. This matches the assignment of land in the different wealth components according to national accounting. The Long Term Housing & Macro Model also endogenizes the split between non-residential wealth into the two components physical capital and non-residential land wealth.[43]

Models that do not capture land as an input in the non-residential sector (Y sector) cannot adequately attribute rising non-residential wealth to rising land prices associated with land scarcity. Rising land prices are, however, of central importance for both the evolution of housing wealth and the evolution of non-residential wealth.

6 Summary

We have presented a novel housing & macro model that is specifically designed to think about long term, time series data on housing and macro variables. The model rests on three premises: (1) fixed overall land endowment; (2) land rivalry between non-residential and residential production; (3) land as an essential input in housing production along the extensive margin (setting up new housing projects), but not along the intensive margin (investing in residential buildings).

To illustrate the model's capabilities, we have applied the model in order to analyze the housing wealth-to-income ratio in four industrialized economies. The model replicates, with remarkable accuracy, the historical evolution of housing wealth (relative to income) after World War II and suggests a further considerable increase. The model also accounts for the close connection of house prices to land prices in the data. The analysis points to the fundamental importance of land for understanding the dynamics of wealth. In line with recent empirical studies, it therefore provides a modern formalization of Ricardo's (1817) principle of land scarcity, which states that economic growth primarily benefits the owners of the fixed factor land.

[43] In standard macroeconomic models without explicit land considerations, K_t is interpreted to capture physical assets including land. This generally appears inappropriate, as land is non-accumulable (Premise 1).

There are many important research questions on the long term macroeconomics of housing, such as those mentioned in the introduction, that could not be discussed so far. We hope that the analytical framework developed in this paper will be applied to address these and related research questions.

7 Appendix

7.1 Wealth-to-Income Ratios

We first define wealth-to-income ratios. We use fraktur (or Gothic) scripture with superscript GDP and NDP to denote wealth-to-GDP and wealth-to-NDP ratios, respectively.

The housing wealth-to-GDP ratio is denoted by $\mathfrak{H}_t^{GDP}$ (speak "fraktur H") and may be expressed as

$$\mathfrak{H}_t^{GDP} \equiv \frac{P_t^H N_t}{GDP_t} = \frac{q_t^N N_t + q_t^X X_t}{GDP_t} = \frac{A_t - K_t}{GDP_t}. \tag{43}$$

The non-residential wealth-to-GDP ratio, denoted by $\mathfrak{N}_t^{GDP}$, is the sum of the ratio of physical capital to GDP, $\mathfrak{K}_t^{GDP} \equiv \frac{K_t}{GDP_t}$, and the ratio of the value of non-residential land (farm land and other productive, non-residential land property) to GDP ("non-residential land wealth-to-GDP ratio"), $\mathfrak{Z}_t^{GDP} \equiv \frac{P_t^Z Z_t^Y}{GDP_t}$, i.e.,

$$\mathfrak{N}_t^{GDP} \equiv \frac{K_t + P_t^Z Z_t^Y}{GDP_t} = \mathfrak{K}_t^{GDP} + \mathfrak{Z}_t^{GDP}. \tag{44}$$

The total (private) wealth-to-GDP ratio, $\mathfrak{W}_t^{GDP}$, reads as

$$\mathfrak{W}_t^{GDP} \equiv \frac{W_t}{GDP_t} = \mathfrak{H}_t^{GDP} + \mathfrak{N}_t^{GDP}. \tag{45}$$

To calculate wealth-to-NDP ratios from wealth-to-GDP ratios, we divide $\mathfrak{H}_t^{GDP}$, $\mathfrak{K}_t^{GDP}$, $\mathfrak{Z}_t^{GDP}$, $\mathfrak{N}_t^{GDP}$, $\mathfrak{W}_t^{GDP}$ by the ratio of net income to gross income ("NDP-to-GDP ratio"), $\iota_t \equiv \frac{NDP_t}{GDP_t}$. We denote wealth-to-NDP ratios by $\mathfrak{H}_t^{NDP} \equiv \frac{\mathfrak{H}_t^{GDP}}{\iota_t}$, $\mathfrak{K}_t^{NDP} \equiv \frac{\mathfrak{K}_t^{GDP}}{\iota_t}$, $\mathfrak{Z}_t^{NDP} \equiv \frac{\mathfrak{Z}_t^{GDP}}{\iota_t}$, $\mathfrak{N}_t^{NDP} \equiv \frac{\mathfrak{N}_t^{GDP}}{\iota_t}$, $\mathfrak{W}_t^{NDP} \equiv \frac{\mathfrak{W}_t^{GDP}}{\iota_t}$.

7.2 Steady State Properties

This section provides analytical results for the long-run equilibrium. We omit the time index t in the remainder of this Appendix.

Proposition 1. (Existence) *Suppose that*

$$\varrho \equiv \frac{\beta}{1-\alpha} - \frac{\gamma(1-\eta)}{1-\gamma\eta} > 0 \tag{A1}$$

holds. Then there exists a unique, non-trivial long run equilibrium in which $D^Y = L^Y/Z^Y$ fulfills

$$D^{Y*} > D\left[= \frac{L}{Z}\right]. \tag{46}$$

Assumption (A1) holds for any reasonable calibration of the model. The allocation of labor and land is characterized by fractions $l^X \equiv L^X/L$, $l^Y \equiv L^Y/L$, $\mathfrak{z}^N \equiv Z^N/Z$, and $\mathfrak{z}^Y \equiv Z^Y/Z$. Equilibrium property (46) is equivalent to $\mathfrak{z}^{Y*} < l^{Y*}$ and $\mathfrak{z}^{N*} > l^{X*}$. The share of land devoted to the housing sector, $\mathfrak{z}^N$, exceeds the share of labor devoted to the housing sector, l^{X*}, whereas the opposite holds in the rest of the economy. The housing sector is endogenously land-intensive, whereas the rest of the economy is labor-intensive.

We next define the "house-price-to-rent ratio" as the ratio of the price of one house with x units of structures put on ψ units of land to the cost of renting h units of housing services produced with the same amount of structures, x. Formally,

$$\mathfrak{p} \equiv \frac{P^H}{ph} = \frac{q^N + q^X x}{ph}. \tag{47}$$

Note that $\mathfrak{p}$ is also equal to the ratio of house capital, $P^H N$, to housing expenditure, pS.

Proposition 2. (Prices). *Under (A1), in long run equilibrium,*

(i) the interest rate is given by

$$r^* = \frac{\rho}{1-\tau_r}; \tag{48}$$

(ii) the price for housing services, p^*, is decreasing in B^X, B^h, increasing in D, B^Y, and independent of ψ;

(iii) the wage rate, w^*, is decreasing in D, independent of B^X, B^h, ψ, and increasing in B^Y;

(iv) the house price-to-rent ratio is

$$\mathfrak{p}^* = \frac{1 + \frac{(1-\gamma)\delta^X}{r^*}}{r^* + \delta^X}; \tag{49}$$

(v) the return to land, R^{Z*}, the land price, P^{Z*}, and the house price per land unit, P^{H*}/ψ, are increasing in D, B^Y, and are independent of B^X, B^h, ψ.

The Keynes-Ramsey rule implies that the long run after-tax interest rate equals the subjective discount rate, $(1-\tau_r)\, r^* = \rho$. An increase in population density, D, means that labor becomes more abundant and land becomes scarcer, in turn lowering the wage rate and raising both the return per unit of land and its price. Consequently, since the housing sector is land-intensive, the price for housing services rises with D. An increase in TFP of the Y sector, B^Y, raises output of the numeraire good for a given factor allocation, thus increasing the relative long run price for housing services, p^*. It also transmits into higher (long run) factor returns, w^*, R^{Z*}, like in standard one-sector models. Higher productivity parameters in the housing sector (increase in B^X or B^h) lower the price of housing services. While the direct effects of productivity improvements on the long run wage rate, w^*, are positive, there is a counteracting and balancing effect on the value of the marginal product of labor in the construction sector through a decrease in the price for housing services. Thus, w^* as well as the long run return to land, R^{Z*}, remain unchanged. In the long run, the land price and the house price change proportionally to the rental rate of land, R^{Z*}, when population density or TFP parameters change.

Proposition 3 (Factor allocation). *Under (A1), in long run equilibrium, the factor allocation as characterized by $\mathfrak{z}^{Y*} = 1 - \mathfrak{z}^{N*}$ and $l^{X*} = 1 - l^{Y*}$ is independent of D, $\mathbf{B}$, ψ.*

Proposition 4 (Wealth-to-GDP ratios and NDP-to-GDP ratio). *Under (A1), in long run equilibrium, the housing wealth-to-GDP ratio is*

$$\mathfrak{H}^{GDP*} = \frac{P^{H*}N^*}{GDP^*} = \frac{1 + \frac{(1-\gamma)\delta^X}{r^*}}{r^* + \left(\frac{r^*+\delta^X}{\theta} + \eta\gamma\delta^X\right)\frac{r^*+\delta^K}{r^*+(1-\alpha)\delta^K} + (1 + (1-\eta)\gamma)\,\delta^X}, \tag{50}$$

the physical-capital-to-GDP ratio is

$$\mathfrak{K}^{GDP*} = \frac{K^*}{GDP^*} = \frac{\alpha}{r^* + \delta^K + \frac{\left(r^*+\delta^X[1+(1-\eta)\gamma]\right)\left(r^*+(1-\alpha)\delta^K\right)}{\frac{r^*+\delta^X}{\theta}+\eta\gamma\delta^X}}, \tag{51}$$

the non-residential land wealth-to-GDP ratio is

$$\mathfrak{Z}^{GDP*} = \frac{P^{Z*}Z^{Y*}}{GDP^*} = \frac{1-\alpha-\beta}{\left(1 + \frac{r^*+(1-\alpha)\delta^K}{r^*+\delta^K}\frac{r^*+\delta^X+(1-\eta)\gamma\delta^X}{\frac{r^*+\delta^X}{\theta}+\eta\gamma\delta^X}\right)r^*}, \tag{52}$$

and the NDP-to-GDP ratio is

$$\iota^* = \frac{NDP^*}{GDP^*} = \frac{1+\frac{1}{\theta}}{1 + \frac{\left(\frac{1}{\theta}+\frac{\eta\gamma\delta^X}{r^*+\delta^X}\right)\left(r^*+\delta^K\right)}{r^*+(1-\alpha)\delta^K} + \frac{(1-\eta)\gamma\delta^X}{r^*+\delta^X}}, \tag{53}$$

Thus, $\mathfrak{H}^{GDP*}$, $\mathfrak{K}^{GDP*}$, $\mathfrak{Z}^{GDP*}$ and ι^* are independent of D, $\mathbf{B}$, ψ.

Corollary 1. *Under (A1), in long run equilibrium, the non-residential wealth-to-GDP ratio, $\mathfrak{N}^{GDP*} = \mathfrak{K}^{GDP*} + \mathfrak{Z}^{GDP*}$, the total wealth-to-GDP ratio, $\mathfrak{W}^{GDP*} = \mathfrak{H}^{GDP*} + \mathfrak{N}^{GDP*}$, and wealth-to-NDP ratios $\mathfrak{H}^{NDP*}$, $\mathfrak{K}^{NDP*}$, $\mathfrak{Z}^{NDP*}$, $\mathfrak{N}^{NDP*}$, $\mathfrak{W}^{NDP*}$ are independent of D, $\mathbf{B}$, ψ.*

Recalling $\mathfrak{H}^{NDP*} = \frac{\mathfrak{H}^{GDP*}}{\iota^*}$, we obtain (32) in the main text from (50) and (53). Proposition 4 and Corollary 1 reflect the result that the long run factor allocation is independent of population density, D, and technology parameters, $\mathbf{B}$ and ψ (Proposition 3). An increase in population density, D, changes the marginal product of labor and the return to land equally in the housing sector and the Y sector, leaving the factor allocation unchanged. Moreover, reflecting homothetic preferences, changes in technological parameters $\mathbf{B}$ and ψ do not induce structural change and leave the factor allocation unaffected as well.

Proposition 5 (Long run amount of structures and housing consumption). *Under (A1), in long run equilibrium,*

(i) the amount of structures per unit of land, X^*/Z*, is increasing in* D*,* B^Y*,* B^X*,* B^h*, and independent of* ψ*;*

(ii) the amount of housing services per capita, S^*/L*, is increasing in* B^Y*,* B^X*,* B^h*, and decreasing in* D*,* ψ*.*

Since the long run land allocation, i.e., $\mathfrak{z}^{N*} = \psi N^*/Z$, is independent of population density, D, and productivity parameters, $\mathbf{B}$, according to Proposition 3, the number of houses per unit of land, N^*/Z, is independent of D and $\mathbf{B}$ as well. Thus, an increase in D or $\mathbf{B}$ means that more structures are built per unit of land, e.g., houses become "higher" rather than more numerous in more densely populated and in more advanced regions, reflecting the opportunity costs of land in its alternative use in the Y sector. Technological progress therefore implies that the amount of housing services increases as well. That would also hold if on each unit of land more houses could be built (decrease in ψ). Finally, because of decreasing returns in transforming structures on a piece of land into housing services, the per capita amount of housing services is decreasing in population density.

Define $H \equiv pNh + wL^X$ as the contribution of the housing sector (housing services and residential construction) to GDP. We finally consider the labor share in total income, $\mathfrak{L} \equiv \frac{wL}{GDP}$, the land income share in the housing sector, $\mathfrak{I} \equiv R^Z Z^N/H$, and the investment (and savings) rate of the economy, $\mathfrak{s} \equiv \frac{\mathcal{I}}{GDP}$. Note that $\mathfrak{s}$ can be decomposed into a non-residential and residential (housing) investment rate, denoted by $\mathfrak{s}^K \equiv \frac{I^K}{GDP}$ and $\mathfrak{s}^H \equiv \frac{I^N + q^X I^X}{GDP}$, respectively, i.e.,

$$\mathfrak{s} = \frac{I^K + I^N + q^X I^X}{GDP} = \mathfrak{s}^K + \mathfrak{s}^H. \tag{54}$$

Proposition 6. (Factor income shares and investment rates) *Under (A1), in long run equilibrium, the labor share in total income is*

$$\mathfrak{L}^* = \frac{w^* L}{GDP^*} = \frac{\beta + \frac{r^*+(1-\alpha)\delta^K}{r^*+\delta^K} \frac{(1-\eta)\gamma\delta^X}{\frac{r^*+\delta^X}{\theta}+\eta\gamma\delta^X}}{1 + \frac{r^*+(1-\alpha)\delta^K}{r^*+\delta^K} \frac{r^*+\delta^X(1+(1-\eta)\gamma)}{\frac{r^*+\delta^X}{\theta}+\eta\gamma\delta^X}}, \tag{55}$$

the land income share in the housing sector is

$$\mathfrak{I}^* = \frac{1-\gamma}{1+\frac{(1-\eta)\gamma\delta^X}{r^*+\delta^X}}. \tag{56}$$

and the economy's total investment rate is

$$\mathfrak{s}^* = 1 - \iota^*. \tag{57}$$

Thus, $\mathfrak{L}^*$, $\mathfrak{I}^*$ and $\mathfrak{s}^*$ are independent of D, $\mathbf{B}$, ψ. The non-residential investment rate, $\mathfrak{s}^{K*}$, and the residential investment rate, $\mathfrak{s}^{H*}$, are independent of D, $\mathbf{B}$, ψ as well.

7.3 Proofs

Proof of Proposition 1 (Existence). We first calculate the partial equilibrium by examining the supply side only, i.e. we take the (relative) price for housing services (p) as well as the interest rate (r) as given. Then we turn to the demand side.

The current-value Hamiltonian of the representative construction firm associated with optimization problem (17) together with the necessary first-order conditions can then be expressed as

$$\mathcal{H}^X \equiv R^X X - M - wL^X + q^X \left[B^X M^\eta \left(L^X\right)^{1-\eta} - \delta^X X \right], \tag{58}$$

$$\left[\frac{\partial \mathcal{H}^X}{\partial M} = \right] - 1 + \eta q^X B^X \left(\frac{L^X}{M}\right)^{1-\eta} = 0, \tag{59}$$

$$\left[\frac{\partial \mathcal{H}^X}{\partial L^X} = \right] - w + (1-\eta) q^X B^X \left(\frac{M}{L^X}\right)^{\eta} = 0, \tag{60}$$

$$\left[-\frac{\partial \mathcal{H}^X}{\partial X} = \right] - R^X + \delta^X q^X = \dot{q}^X - r q^X. \tag{61}$$

Note that (18) follows from (61).

The typical final output firm maximizes profits given by

$$\Pi = B^Y \left(K^Y\right)^{\alpha} \left(L^Y\right)^{\beta} (Z^Y)^{1-\alpha-\beta} - \left(r + \delta^K\right) K^Y - wL^Y - R^Z Z^Y. \tag{62}$$

Using $K^Y = K$, the necessary first-order conditions are

$$\left[\alpha \frac{Y}{K^Y} = \right] \alpha B^Y K^{\alpha-1} \left(L^Y\right)^{\beta} (Z^Y)^{1-\alpha-\beta} = r + \delta^K, \tag{63}$$

$$\left[\beta \frac{Y}{L^Y} = \right] \beta B^Y K^{\alpha} \left(L^Y\right)^{\beta-1} (Z^Y)^{1-\alpha-\beta} = w, \tag{64}$$

$$\left[(1-\alpha-\beta) \frac{Y}{Z^Y} = \right] (1-\alpha-\beta) B^Y K^{\alpha} \left(L^Y\right)^{\beta} (Z^Y)^{-\alpha-\beta} = R^Z. \tag{65}$$

Combining (63) and (64), leads to

$$K = \frac{\alpha}{\beta} \frac{w}{r + \delta^K} L^Y. \tag{66}$$

Substituting (66) into (63) and (65), we obtain

$$w = \beta \left(\frac{\alpha}{r + \delta^K} \right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left(\frac{Z^Y}{L^Y} \right)^{\frac{1-\alpha-\beta}{1-\alpha}}, \tag{67}$$

$$R^Z = (1-\alpha-\beta) B^Y \left(\frac{\alpha}{\beta} \frac{w}{r + \delta^K} \right)^{\alpha} \left(\frac{L^Y}{Z^Y} \right)^{\alpha+\beta}. \tag{68}$$

respectively. Setting $\dot{N} = 0$ in (15) implies, for the long run,

$$q^N = \psi P^Z. \tag{69}$$

Substituting (13) into (16) and using $\dot{q}^N = 0$, we have

$$q^N = \frac{(1-\gamma) p B^h x^{\gamma}}{r}. \tag{70}$$

Setting $\dot{P}^Z = \dot{q}^X = 0$ in (21) and (18), respectively, we obtain, for the long run,

$$P^Z = \frac{R^Z}{r}, \tag{71}$$

$$q^X = \frac{R^X}{r + \delta^X}. \tag{72}$$

Setting $\dot{X} = 0$ in (6) and using $X = Nx$, we have

$$x = \frac{B^X M^\eta \left(L^X\right)^{1-\eta}}{\delta^X N}, \tag{73}$$

Next, combine (59) with (72) to obtain

$$M = L^X \left(\frac{\eta R^X B^X}{r + \delta^X} \right)^{\frac{1}{1-\eta}}. \tag{74}$$

Now substitute (73) into (12) to find

$$R^X = pB^h \gamma \left(\frac{B^X M^\eta \left(L^X\right)^{1-\eta}}{\delta^X N} \right)^{\gamma - 1}. \tag{75}$$

Substituting (75) into (74) yields a useful expression for M:

$$M = \left(L^X\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{\gamma \eta p B^h \left(\delta^X N\right)^{1-\gamma} \left(B^X\right)^\gamma}{r + \delta^X} \right)^{\frac{1}{1-\gamma\eta}}. \tag{76}$$

Combining (60) with (72) and (74) we obtain

$$w = (1-\eta)\eta^{\frac{\eta}{1-\eta}} \left(\frac{R^X B^X}{r + \delta^X} \right)^{\frac{1}{1-\eta}}. \tag{77}$$

Substituting (75) into (77) leads to

$$w = (1-\eta)\eta^{\frac{\eta}{1-\eta}} \left(\frac{\gamma p B^h}{r + \delta^X} \right)^{\frac{1}{1-\eta}} \left(B^X\right)^{\frac{\gamma}{1-\eta}} \left(\delta^X N\right)^{\frac{1-\gamma}{1-\eta}} M^{\frac{\eta(\gamma-1)}{1-\eta}} \left(L^X\right)^{\gamma-1}. \tag{78}$$

Substituting (76) into (78) gives us a useful expression for w:

$$w = (1-\eta)\eta^{\frac{\gamma\eta}{1-\gamma\eta}} \left(\frac{\gamma B^h p}{r+\delta^X}\right)^{\frac{1}{1-\gamma\eta}} \left(B^X\right)^{\frac{\gamma}{1-\gamma\eta}} \left(\frac{N\delta^X}{L^X}\right)^{\frac{1-\gamma}{1-\gamma\eta}}. \tag{79}$$

Combining (79) with (67) leads to

$$\begin{aligned} \frac{L^X}{N} &= \frac{(1-\eta)^{\frac{1-\gamma\eta}{1-\gamma}} \eta^{\frac{\gamma\eta}{1-\gamma}} \left(\frac{\gamma B^h p}{r+\delta^X}\right)^{\frac{1}{1-\gamma}} \left(B^X\right)^{\frac{\gamma}{1-\gamma}} \delta^X}{\left[\beta \left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha}{1-\alpha}} \left(B^Y\right)^{\frac{1}{1-\alpha}}\right]^{\frac{1-\gamma\eta}{1-\gamma}}} \left(\frac{L^Y}{Z^Y}\right)^{\frac{1-\alpha-\beta}{1-\alpha}\frac{1-\gamma\eta}{1-\gamma}} \\ &\equiv \Phi\left(\frac{L^Y}{Z^Y}, p, r, \mathbf{B}\right), \end{aligned} \tag{80}$$

Note that Φ is increasing as a function of L^Y/Z^Y. Moreover, Φ is strictly concave as a function of L^Y/Z^Y if and only if (A1) holds. Substituting (70) and (71) into (69), we obtain

$$[\pi =]\, (1-\gamma)pB^h x^\gamma = \psi R^Z. \tag{81}$$

Inserting (73) and (68) into (81), we get

$$\frac{(1-\gamma)pB^h}{w} \left(\frac{B^X M^\eta \left(L^X\right)^{1-\eta}}{\delta^X N}\right)^\gamma = \frac{\psi\,(1-\alpha-\beta)\,B^Y}{w^{1-\alpha}} \left(\frac{\alpha}{\beta}\frac{1}{r+\delta^K}\right)^\alpha \left(\frac{L^Y}{Z^Y}\right)^{\alpha+\beta}. \tag{82}$$

Substituting (67) and (78) into the right-hand side and left-hand side of (82), respectively, yields

$$\frac{(1-\gamma)M^{\frac{\eta(1-\gamma\eta)}{1-\eta}} \left(L^X\right)^{1-\eta\gamma}}{(1-\eta)\left(\eta p B^h\right)^{\frac{\eta}{1-\eta}} \left(\frac{\gamma}{r+\delta^X}\right)^{\frac{1}{1-\eta}} \left(B^X\right)^{\frac{\gamma\eta}{1-\eta}} \left(\delta^X N\right)^{\frac{1-\gamma\eta}{1-\eta}}} = \frac{\psi\,(1-\alpha-\beta)}{\beta}\frac{L^Y}{Z^Y}. \tag{83}$$

Now substitute (76) into (83) to find

$$\frac{L^X}{N} = \psi\upsilon(r)\frac{L^Y}{Z^Y}, \text{ with} \tag{84}$$

$$\upsilon(r) \equiv \frac{(1-\alpha-\beta)\,(1-\eta)\gamma\delta^X}{\beta(1-\gamma)(r+\delta^X)}. \tag{85}$$

Finally, combine the right-hand sides of (80) and (84) to obtain a unique, nontrivial solution for $D^Y = L^Y/Z^Y$ that is given by

$$D^Y = \left(\frac{\eta^{\frac{\gamma\eta}{1-\gamma\eta}} \left(\frac{B^X\gamma}{r+\delta^X}\right)^{\frac{\gamma}{1-\gamma\eta}} \left(B^h p\right)^{\frac{1}{1-\gamma\eta}} \left(\frac{1-\eta}{\beta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}}}{\left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha}{1-\alpha}} \left(B^Y\right)^{\frac{1}{1-\alpha}} \left(\frac{\psi(1-\alpha-\beta)}{1-\gamma}\right)^{\frac{1-\gamma}{1-\gamma\eta}}} \right)^{\frac{1}{\varrho}} \equiv \tilde{D}^Y(p, r, \mathbf{B}). \tag{86}$$

According to (84) and $L^Y/Z^Y = \tilde{D}^Y(p, r, \mathbf{B})$, we have

$$\frac{L^X}{N} = \psi \upsilon(r) \tilde{D}^Y(p, r, \mathbf{B}). \tag{87}$$

Next, using $L^Y = D^Y Z^Y$ and $L^X = N\psi\upsilon(r)D^Y$, $L^X + L^Y = L$ and $Z^Y = Z - \psi N$ we obtain

$$\frac{N}{Z} = \frac{1}{\psi\left[1 - \upsilon(r)\right]} \left(1 - \frac{D}{\tilde{D}^Y(p, r, \mathbf{B})}\right), \tag{88}$$

$$\mathfrak{z}^Y = \frac{Z^Y}{Z} = \frac{1}{1 - \upsilon(r)} \left(\frac{D}{\tilde{D}^Y(p, r, \mathbf{B})} - \upsilon(r)\right), \tag{89}$$

$$\frac{N}{Z^Y} = \frac{1}{\psi} \frac{\frac{\tilde{D}^Y(p,r,\mathbf{B})}{D} - 1}{1 - \upsilon(r)\frac{\tilde{D}^Y(p,r,\mathbf{B})}{D}}. \tag{90}$$

From $L^X = N\psi\upsilon(r)D^Y$, $D = L/Z$ and (88), we also have

$$\frac{L^X}{L} = \frac{\upsilon(r)}{1 - \upsilon(r)} \left(\frac{\tilde{D}^Y(p, r, \mathbf{B})}{D} - 1\right). \tag{91}$$

Thus, as $L^Y/L = 1 - L^X/L$, we obtain

$$\frac{L^Y}{L} = \frac{1}{1 - \upsilon(r)} \left(1 - \upsilon(r)\frac{\tilde{D}^Y(p, r, \mathbf{B})}{D}\right). \tag{92}$$

This closes the supply side analysis.

According to (27) and (70), we can write for the long run:

$$q^N N = \frac{(1 - \gamma)pS}{r}. \tag{93}$$

Moreover, combining (12) with (27), we have $R^X Nx = \gamma pS$. Combining $X = Nx$, (72) and (73) we thus find that for the long run:

$$q^X X = \frac{R^X Nx}{r + \delta^X} = \frac{\gamma pS}{r + \delta^X}. \tag{94}$$

Using (66), (93) and (94) in (9) leads to long run total asset value:

$$A = \frac{\alpha}{\beta}\frac{wL^Y}{r + \delta^K} + \frac{(1-\gamma)pS}{r} + \frac{\gamma pS}{r + \delta^X}. \tag{95}$$

Next, substituting (73) into condition (27) and using (76) gives us

$$pS = N\left(B^h p\right)^{\frac{1}{1-\gamma\eta}}\left(B^X\right)^{\frac{\gamma}{1-\gamma\eta}}\left(\frac{L^X}{N}\frac{1}{\delta^X}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}}\left(\frac{\gamma\eta}{r+\delta^X}\right)^{\frac{\gamma\eta}{1-\gamma\eta}}. \tag{96}$$

The total tax revenue that is redistributed to households reads as

$$T = \tau_r r W. \tag{97}$$

As $W = A + P^Z Z^Y$, we have $\dot{W} = \dot{A} + \dot{P}^Z Z^Y + P^Z \dot{Z}^Y$. In steady state, $\dot{P}^Z = \dot{Z}^Y = 0$, thus, $\dot{W} = \dot{A}$.

According to (8) and (11), the current-value Hamiltonian for the household optimization problem (equilibrium condition 1 in Definition 1) is given by

$$\mathcal{H} \equiv \log C + \theta \log S + \lambda\left[(1-\tau_r)rW + wL + T - C - pS\right], \tag{98}$$

where λ is the multiplier (co-state variable) associated with financial asset holding, A. Necessary optimality conditions are $\partial\mathcal{H}/\partial C = \partial\mathcal{H}/\partial S = 0$ (control variables), $\dot{\lambda} = \rho\lambda - \partial\mathcal{H}/\partial W$ (co-state variable), and the corresponding transversality condition. Thus,

$$\lambda = \frac{1}{S}, \tag{99}$$

$$\frac{\theta}{S} = \lambda p, \tag{100}$$

$$\frac{\dot{\lambda}}{\lambda} = \rho - (1 - \tau_r)r. \tag{101}$$

Combining (99) and (100), we have

$$C = \frac{pS}{\theta}, \tag{102}$$

whereas combining (99) and (101) yields the Keynes-Ramsey rule

$$\frac{\dot{C}}{C} = (1 - \tau_r)r - \rho. \tag{103}$$

We seek for a steady state without long run growth. Setting $\dot{C} = 0$ in (103) gives us the long run interest rate

$$r^* = \frac{\rho}{1 - \tau_r}. \tag{104}$$

Combining (102) and (11) with $\dot{W} = 0$ and using (97) we obtain

$$\left(\frac{1}{\theta} + 1\right) pS = rW + wL. \tag{105}$$

Substituting $W = A + P^Z Z^Y$ into (105) and using (95) and $R^Z = rP^Z$ leads to

$$\left(\frac{1}{\theta} + \frac{\gamma\delta^X}{r + \delta^X}\right) pS = w\left(\frac{\alpha}{\beta}\frac{rL^Y}{r + \delta^K} + L\right) + R^Z Z^Y. \tag{106}$$

Inserting (67) into (68) leads to

$$R^Z = (1 - \alpha - \beta)\left(\frac{\alpha}{r + \delta^K}\right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left(\frac{L^Y}{Z^Y}\right)^{\frac{\beta}{1-\alpha}}. \tag{107}$$

By substituting (67), (96) and (107) into (106) we obtain

$$\begin{aligned} &\left(\frac{1}{\theta} + \frac{\gamma\delta^X}{r + \delta^X}\right) \frac{N}{Z^Y} \left(B^h p\right)^{\frac{1}{1-\gamma\eta}} \left(B^X\right)^{\frac{\gamma}{1-\gamma\eta}} \left(\frac{L^X}{N}\frac{1}{\delta^X}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{\gamma\eta}{r + \delta^X}\right)^{\frac{\gamma\eta}{1-\gamma\eta}} \left(\frac{L^Y}{Z^Y}\right)^{-\frac{\beta}{1-\alpha}} \\ = &\left(\frac{\alpha}{r + \delta^K}\right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left[\frac{\alpha r}{r + \delta^K} + \beta\left(\frac{L}{L^Y} - 1\right) + 1 - \alpha\right]. \end{aligned} \tag{108}$$

Substituting $L^Y/Z^Y = \tilde{D}^Y(p, r, \mathbf{B})$, (87) and (90) into (108) and recalling the definition of ϱ in (A1) implies, for the long run,

$$\frac{\left(\frac{1}{\theta} + \frac{\gamma\delta^X}{r^*+\delta^X}\right)\left(1 - \frac{\tilde{D}^Y(p^*,r^*,\mathbf{B})}{D}\right)\left(B^h p^*\right)^{\frac{1}{1-\gamma\eta}}\left(B^X\right)^{\frac{\gamma}{1-\gamma\eta}}\left(\frac{\upsilon(r^*)}{\delta^X}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}}\left(\frac{\gamma\eta B^M}{r^*+\delta^X}\right)^{\frac{\gamma\eta}{1-\gamma\eta}}}{\psi^{\frac{1-\gamma}{1-\gamma\eta}}\left(\upsilon(r^*)\frac{\tilde{D}^Y(p^*,r^*,\mathbf{B})}{D} - 1\right)\tilde{D}^Y(p^*, r^*, \mathbf{B})^{\varrho}}$$

$$= \left(\frac{\alpha}{r^* + \delta^K}\right)^{\frac{\alpha}{1-\alpha}}\left(B^Y\right)^{\frac{1}{1-\alpha}}\left[\frac{\alpha r^*}{r^* + \delta^K} + \frac{\beta\upsilon(r^*)\left(1 - \frac{\tilde{D}^Y(p^*,r^*,\mathbf{B})}{D}\right)}{\upsilon(r^*)\frac{\tilde{D}^Y(p^*,r^*,\mathbf{B})}{D} - 1} + 1 - \alpha\right]. \quad (109)$$

Substituting (85) and (86) in (109) we find that the long run price for housing services, $p^* \equiv \tilde{p}^*(r^*, \mathbf{B}, D)$, is implicitly given by

$$\tilde{D}^Y(p^*, r^*, \mathbf{B}) = \frac{D}{\mu(r^*)} \equiv D^{Y*} \text{ with } \mu(r) \equiv \frac{\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{r+\delta^X} + \frac{r+(1-\alpha)\delta^K}{r+\delta^K}\frac{(1-\eta)\gamma\delta^X}{\beta\left(r+\delta^X\right)}}{\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{r+\delta^X} + \frac{1-\gamma}{1-\alpha-\beta}\frac{r+(1-\alpha)\delta^K}{r+\delta^K}}. \quad (110)$$

We have $\tilde{D}^Y(0, r^*, \mathbf{B}) = 0$, $\lim_{p\to\infty}\tilde{D}^Y(p, r^*, \mathbf{B}) \to \infty$ and $\frac{\partial\tilde{D}^Y}{\partial p} > 0$, according to (86) and (A1). Thus, $p^* > 0$ exists and is unique. Moreover, is easy to show that $\mu(r) \in (\upsilon(r), 1)$ if and only if $\upsilon(r) < 1$, which holds for all r according to (A1).[44] Thus, also (46) holds under (A1). This concludes the proof. ■

Proof of Proposition 2 (Prices). Part (i) follows from (104). Using (86) and applying the implicit function theorem to (110) confirms part (ii). To prove parts (iii)-(v), first use (67) to find

$$w^* = \beta\left(\frac{\alpha}{r^* + \delta^K}\right)^{\frac{\alpha}{1-\alpha}}\left(B^Y\right)^{\frac{1}{1-\alpha}}\left(D^{Y*}\right)^{-\frac{1-\alpha-\beta}{1-\alpha}}. \quad (111)$$

[44] Use that $\upsilon(r) < 1$ is equivalent to

$$\frac{\beta}{1-\alpha} - \frac{\gamma(1-\eta)}{1-\gamma\eta}\underbrace{\frac{\delta^X}{\frac{(1-\gamma)r}{1-\gamma\eta} + \delta^X}}_{<1} > 0.$$

Moreover, recall from (71) that

$$P^{Z*} = \frac{R^{Z*}}{r^*} \text{ with } R^{Z*} = (1-\alpha-\beta)\left(\frac{\alpha}{r^*+\delta^K}\right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left(D^{Y*}\right)^{\frac{\beta}{1-\alpha}}, \tag{112}$$

according to (107). Next, use (4), (12), (104), (70) and (72) in (47) to confirm (49). Combining (70) and (121) and using (4) yields

$$p^* h^* = \frac{\psi R^{Z*}}{1-\gamma}. \tag{113}$$

Using (49) and (113) in $P^{H*} = p^* h^* \mathfrak{p}^*$, we obtain

$$P^{H*} = \frac{\psi (1-\alpha-\beta)\left(\frac{\alpha}{r^*+\delta^K}\right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left(D^{Y*}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{1}{1-\gamma} + \frac{\delta^X}{r^*}\right)}{r^* + \delta^X}. \tag{114}$$

Using that, according to (110), D^{Y*} is independent of $\mathbf{B}$ and proportional to D confirms parts (iii)-(v). This concludes the proof. ■

Proof of Proposition 3 (Factor allocation). Combining (89) and (91) with (110), we obtain

$$\mathfrak{z}^{Y*} = \frac{Z^{Y*}}{Z} = \frac{\mu(r^*) - \upsilon(r^*)}{1-\upsilon(r^*)} [= 1 - \mathfrak{z}^{N*}], \tag{115}$$

$$l^{X*} = \frac{L^{X*}}{L} = \frac{\upsilon(r^*)}{1-\upsilon(r^*)} \frac{1-\mu(r^*)}{\mu(r^*)} [= 1 - l^{Y*}], \tag{116}$$

respectively. Recalling (104) as well as the definitions of $\upsilon(r)$ and $\mu(r)$ in (85) and (110), respectively, concludes the proof. ■

Proof of Proposition 4 (Wealth-to-GDP ratios and NDP-to-GDP ratio). According to (88) and (110), we have

$$\frac{N^*}{Z} = \frac{1-\mu(r^*)}{\psi\left[1-\upsilon(r^*)\right]}. \tag{117}$$

Using (29), we can write the housing wealth-to-GDP ratio, the physical capital-to-GDP ratio, and the non-residential land-wealth-to-GDP ratio as

$$\mathfrak{H}^{GDP} = \frac{P^H N}{GDP} = \frac{\mathfrak{p}}{1 + \frac{Y}{pNh} + \frac{wL^X}{phN}}, \tag{118}$$

$$\mathfrak{K}^{GDP} = \frac{K}{GDP} = \frac{\frac{K}{Y}}{1 + \frac{pNh}{Y} + \frac{wL}{Y}\frac{L^X}{L}}, \tag{119}$$

$$\mathfrak{Z}^{GDP} = \frac{P^Z Z^Y}{GDP} = \frac{\frac{P^Z Z^Y}{Y}}{1 + \frac{pNh}{Y} + \frac{wL}{Y}\frac{L^X}{L}}, \tag{120}$$

respectively. According to (69) and (71), in the long run,

$$q^{N*} = \frac{\psi R^{Z*}}{r^*}. \tag{121}$$

Substituting the expression for R^{Z*} as given by (112) into (113) leads to

$$p^* h^* = \frac{\psi (1 - \alpha - \beta)}{1 - \gamma} \left(\frac{\alpha}{r^* + \delta^K} \right)^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}} \left(D^{Y*}\right)^{\frac{\beta}{1-\alpha}}. \tag{122}$$

According to (1), (4) and (66), we have

$$\frac{Y}{pNh} = \frac{B^Y \left(\frac{\alpha}{\beta} \frac{w}{r+\delta^K} \right)^{\alpha} \left(\frac{L^Y}{Z^Y} \right)^{\alpha+\beta} \frac{Z^Y}{Z}}{ph\frac{N}{Z}}. \tag{123}$$

Moreover, using the definitions of $\upsilon(r)$ and $\mu(r)$ in (85) and (110), respectively, we find

$$\frac{\mu(r^*) - \upsilon(r^*)}{1 - \mu(r^*)} = \frac{1 - \alpha - \beta}{1 - \gamma} \frac{\left(\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{r^* + \delta^X} \right) (r^* + \delta^K)}{r^* + (1 - \alpha)\delta^K}. \tag{124}$$

Substituting (111), (115), (117), (122) and $L^{Y*}/Z^{Y*} = D^{Y*}$ into (123), and using (124), we get

$$\frac{Y^*}{p^* N^* h^*} = \frac{\left(\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{r^* + \delta^X} \right) (r^* + \delta^K)}{r^* + (1 - \alpha)\delta^K} \tag{125}$$

in long run equilibrium. Moreover,

$$\frac{wL^X}{phN} = \frac{w}{ph} l^X \frac{L/Z}{N/Z}. \tag{126}$$

Substituting (111), (116) and (117) into (126) and using (104), $D = L/Z$ and $\upsilon(r^*)$ as given by (85), we obtain

$$\frac{w^* L^{X*}}{p^* N^* h^*} = \frac{(1-\eta)\gamma\delta^X}{r^* + \delta^X}. \tag{127}$$

Using (49), (125) and (127) in (118) confirms (50).

According to (63), (64), (65) and $L^Y = L(1 - l^X)$, we have

$$\frac{K}{Y} = \frac{\alpha}{r + \delta^K}, \tag{128}$$

$$\frac{wL}{Y} = \frac{\beta}{1 - l^X}, \tag{129}$$

$$\frac{R^Z Z^Y}{Y} = 1 - \alpha - \beta. \tag{130}$$

Using that $R^{Z*} = r^* P^{Z*}$, in long run equilibrium,

$$\frac{P^{Z*} Z^{Y*}}{Y^*} = \frac{1 - \alpha - \beta}{r^*}. \tag{131}$$

Substituting (125), (128), (129) and (131) in (119) and (120), respectively, and using (85), (116) and (124) confirm (51) and (52).

Finally, in the long run, investment equals depreciation. Thus, in the long run, NDP equals consumption expenditure, $NDP^* = \mathcal{C}^*$. The ratio of consumption expenditure to GDP can be written as

$$\iota^* = \frac{NDP^*}{GDP^*} = \frac{\mathcal{C}^*}{GDP^*} = \frac{C^* + p^* S^*}{GDP^*} = \frac{\frac{C^*}{p^* S^*} + 1}{1 + \frac{Y^*}{p^* N^* h^*} + \frac{w^* L^{X*}}{p^* N^* h^*}}. \tag{132}$$

Using (102), (125) and (127) in (132) confirms (53). This concludes the proof. ■

Proof of Corollary 1. Results immediately follow from Proposition 4, (44) and

(45). ■

Proof of Proposition 5 (Long run amount of structures and housing consumption). Inserting (76) into (73) and using (87), (117) and (110) implies that the long run equilibrium amount of structure per unit of land is given by

$$\frac{X^*}{Z} = \frac{1-\mu(r)}{1-\upsilon(r^*)}\left(\frac{\upsilon(r^*)}{\delta^X}\frac{D}{\mu(r^*)}\right)^{\frac{1-\eta}{1-\gamma\eta}}\left(\frac{\gamma\eta B^h p^*}{(r^*+\delta^X)\psi^{1-\gamma}}\right)^{\frac{\eta}{1-\gamma\eta}}\left(B^X\right)^{\frac{1}{1-\gamma\eta}}. \tag{133}$$

According to (86) and (110), we have

$$B^h p^* = \left(\frac{D}{\mu(r^*)}\right)^{\varrho(1-\gamma\eta)}\frac{\left(\frac{\alpha}{r^*+\delta^K}\right)^{\frac{\alpha(1-\gamma\eta)}{1-\alpha}}\left(B^Y\right)^{\frac{1-\gamma\eta}{1-\alpha}}\left(\frac{\psi(1-\alpha-\beta)}{1-\gamma}\right)^{1-\gamma}}{\eta^{\gamma\eta}\left(\frac{B^X\gamma}{r^*+\delta^X}\right)^{\gamma}\left(\frac{1-\eta}{\beta}\right)^{\gamma(1-\eta)}} \tag{134}$$

Substituting (134) into (133) and using the definition of ϱ in (A1) yields

$$\frac{X^*}{Z} = \chi(r^*)D^{1-\eta+\frac{\beta\eta}{1-\alpha}}B^X\left(B^Y\right)^{\frac{\eta}{1-\alpha}}, \text{ where} \tag{135}$$

$$\chi(r) \equiv \frac{\eta^\eta\left(\frac{\beta}{1-\eta}\right)^{\frac{\gamma\eta(1-\eta)}{1-\gamma\eta}}\left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha\eta}{1-\alpha}}\left(\frac{\upsilon(r)}{\delta^X}\right)^{\frac{1-\eta}{1-\gamma\eta}}[1-\mu(r)]\left(\frac{1-\alpha-\beta}{1-\gamma}\frac{\gamma}{r+\delta^X}\right)^{\frac{\eta(1-\gamma)}{1-\gamma\eta}}}{[1-\upsilon(r)]\mu(r)^{1-\eta+\frac{\beta\eta}{1-\alpha}}}. \tag{136}$$

According to (27) and $X = Nx$, the consumption of housing services per capita is given by

$$\frac{S}{L} = \frac{N}{L}B^h\left(\frac{X}{N}\right)^\gamma = \left(\frac{N}{Z}\right)^{1-\gamma}\frac{B^h}{D}\left(\frac{X}{Z}\right)^\gamma. \tag{137}$$

Using (117) and (133) in (137), we obtain

$$\frac{S^*}{L} = \chi(r^*)^\gamma\left(\frac{1-\mu(r^*)}{\psi(1-\upsilon(r^*))}\right)^{1-\gamma}\frac{B^h\left(B^X\right)^\gamma\left(B^Y\right)^{\frac{\gamma\eta}{1-\alpha}}}{D^{1-\gamma+\frac{(1-\alpha-\beta)\gamma\eta}{1-\alpha}}}. \tag{138}$$

Parts (i) and (ii) follow from (135) and (138), respectively, using that $\upsilon(r^*)$, $\mu(r^*)$ and $\chi(r^*)$ are independent of D, $\mathbf{B}$ and ψ. This concludes the proof. ■

Proof of Proposition 6 (Factor income shares and investment rates). Using

(29), we have

$$\mathfrak{L} = \frac{wL}{GDP} = \frac{\frac{wL}{Y}}{1 + \frac{pNh}{Y} + \frac{wL}{Y}\frac{L^X}{L}}. \tag{139}$$

Using (116) and (129) together with (85), (104) and (110) in (139) confirms (55). Moreover, using $H = pNh + wL^X$, we have

$$\mathfrak{I} = \frac{R^Z Z^N}{H} = \frac{R^Z Z^N}{pNh + wL^X}. \tag{140}$$

For the long run, according to (140), we can write

$$\mathfrak{I}^* = \frac{R^{Z*}\mathfrak{z}^{N*}}{p^* h^* \frac{N^*}{Z} + w^* l^{X*} D}, \tag{141}$$

where we used $\mathfrak{z}^N = Z^N/Z$, $l^X = L^X/Z$ and $D = L/Z$. Using (111), (112), (113), (115), (116) and (117) in (141) and employing (85) confirms (56). Finally, since total investment is equal to capital depreciation in the long run, $\mathcal{I}^* = \delta^K K^* + \delta^X q^{X*} X^*$,

$$\mathfrak{s}^* = \frac{\mathcal{I}^*}{GDP^*} = 1 - \frac{NDP^*}{GDP^*}, \tag{142}$$

confirming (57). Note that $\dot{K} = 0$ implies $I^{K*} = \delta^K K^*$; thus, $s^{K*} = \frac{I^{K*}}{GDP^*} = \delta^K \mathfrak{K}^{GDP*}$, confirming its claimed properties by recalling Proposition 4. Finally, recall $s^{H*} = s^* - s^{K*}$. This concludes the proof. ■

7.4 Calibration

The set of country-specific, time-invariant parameters comprises the capital income tax rate, τ_r, and the initial conditions, K_0, N_0 and X_0. The set of country-specific, time-varying parameters comprises the population density, D, and TFP parameters, B^Y and B^X. (We normalize $B^h = 1$.) The remaining parameters are viewed as being general and set to match the relevant empirical characteristics of the US economy. The calibration strategy does *not* assume that the US currently is in long run equilibrium.

According to (102), the marginal rate of substitution between the two consumption goods equals the relative price, $p = \theta C/S$. Data on housing expenditures for the US indicate that the ratio of households' housing expenditures to total consumption expenditures,

$$\frac{pS}{\mathcal{C}} = \frac{pS}{C + pS} = \frac{\theta}{1+\theta}, \tag{143}$$

is quite stable over time for the period 1960-2012 and equals, on average, about 18 percent (Knoll et al., 2016). The value is very close to the average values for the UK, France and Germany. Setting the expression in (143) to 0.18 suggests $\theta = 0.22$.

We next turn to depreciation rates (δ^X and δ^K). For the housing sector (residential structures), Hornstein (2009, p. 13) suggests, by referring to data from the US Bureau of Economic Analysis (2004), that $\delta^X = 0.015$. The depreciation rate of physical capital, δ^K, can be inferred from the definition of gross investment in physical capital, $I^K = \dot{K} + \delta^K K$, i.e.,

$$\delta^K = \frac{I^K}{K} - \frac{\dot{K}}{K}. \tag{144}$$

We assume that, off-steady state, physical capital investment is 10 percent of the physical capital stock, $I^K/K = 0.1$.[45] Assuming that the average annual growth rate of physical capital, $\dot{K}/K$, was about three percent (the sum of the long term GDP per capita growth rate of two percent and the population growth rate of one percent), we arrive at $\delta^K = 0.07$, according to (144).[46]

Turning to concavity parameters of the production function in the Y sector, we start by noting that β equals, in equilibrium, the expenditure share for labor in the Y sector, i.e., $\beta = wL^Y/Y$, according to (64). Recall that $H = pNh + wL^X$ is the value-added of the housing sector (housing services and residential construction). According to (29), we

[45] The average ratio of US non-residential investment to GDP for the period 1969-2014 amounts to 12.6 percent (Bureau of Economic Analysis, 2015a, Tab. 1.1.10), with little variation over time. Consequently, suppose $\mathfrak{s}^K = I^K/GDP = 0.126$. According to Piketty and Zucman (2014b, Tab. US.6c), in the US, the non-residential wealth-to-NDP ratio was $\mathfrak{N}^{NDP} \approx 2$ (e.g., in the period 1960-2010, 1970-2010 or 1980-2010). Unfortunately, the data does not allow us to decompose non-residential wealth into physical capital and land. Accounting for depreciation, it is reasonable to assume that the physical capital-to-GDP ratio, $\mathfrak{K}^{GDP} < \mathfrak{K}^{NDP} = \mathfrak{N}^{NDP} - \mathfrak{Z}^{NDP}$, is somewhere between 100-150 percent. Assuming $\mathfrak{K}^{GDP} = 1.26$, we arrive at $I^K/K = \mathfrak{s}^K/\mathfrak{K}^{GDP} = 0.1$.

[46] The value also seems reasonable according to the evidence on depreciation rates for 36 manufacturing sectors, reported in House and Shapiro (2008).

can thus write $GDP = Y + H$. Using $L^Y/L = 1 - l^X$, $Y = GDP - H$, and recalling that the labor share in GDP is $\mathfrak{L} = \frac{wL}{GDP}$, we get

$$\beta = \frac{(1 - l^X)\mathfrak{L}}{1 - \frac{H}{GDP}}. \tag{145}$$

According to Bureau of Economic Analysis (2015b), the average value-added of the housing sector and (residential and non-residential) construction as percentage of GDP in the period 1998-2001 (prior to the housing boom) in the US was, on average, 9.1 percent and 4.4 percent, respectively. In our model, $q^X I^X$ is the value-added of residential construction. The ratio of residential to total investment in structures during 1998-2001 was 61 percent (calculated from Bureau of Economic Analysis, 2015c), which suggest a value-added of residential construction relative to GDP, $\frac{q^X I^X}{GDP}$, of $0.61 \times 4.4 = 2.7$ percent. Thus, we set $\frac{H}{GDP}$ to $9.1 + 2.7 \approx 12$ percent. According to Henderson (2015, Tab. 2.1), the US employment share in construction decreased from 4.8 percent in 2004 (before the financial crises) to 4.1 percent in 2014 (after the financial crises). Taking an intermediate value of 4.5 percent and multiplying it by the fraction of residential investment in total investment in structures (61 percent), we arrive at $l^X = 0.045 \times 0.61 = 0.027$. According to Karabarbounis and Neiman (2014, "CLS KN merged"), the corporate US labor share in total income was pretty stable in the period 1975-2008, only recently declining in a more pronounced way. The average value for the period 1975-2012 was 62 percent. Using $\frac{H}{GDP} = 0.12$, $l^X = 0.027$ and $\mathfrak{L} = 0.62$, we arrive at $\beta = 0.973 \times 0.62/0.88 \approx 0.69$, according to (145).

According to (4), (13) and (27), the ratio of housing services producers' profits to their revenue is $\frac{\pi}{ph} = 1 - \gamma$. We approximate $N\pi$ with the average for the period 1995-2006 of "output of housing services" minus costs minus the "current surplus of government enterprises" (mostly mortgage finance agencies) and divide it by the "output of housing services" (a measure for Nph).[47] We arrive at $\gamma = 0.9$.

[47] The data is taken from http://www.bea.gov/national/pdf/output_rip_facq.pdf. Mayerhauser and Reinsdorf (2007) provide a discussion.

We next derive the numerical value of the output elasticity of materials in the construction sector, η. Combining the first-order condition with respect to labor in the construction sector (60) with (5), we have $wL^X = (1-\eta)q^X I^X$. Using $wL^X = l^X \mathfrak{L} GDP$, we obtain

$$\eta = 1 - \frac{wL^X}{q^X I^X} = 1 - \frac{l^X \mathfrak{L}}{q^X I^X / GDP}. \tag{146}$$

Recalling $\frac{q^X I^X}{GDP} = 0.027$, $\mathfrak{L} = 0.62$, and $l^X = 0.027$, we obtain $\eta = 0.38$.

The long run interest rate, $r^* = \frac{\rho}{1-\tau_r}$, plays an important role for long run asset prices, including the land price. Albeit not modelled here explicitly, the return to equity contains a risk premium; also, ownership of firms and land is at risk of devaluation because of natural disaster, environmental damage or expropriation by government action. Thus, we shall set the subjective discount rate at a value that is in the middle or at the upper end of the typical range used in calibration exercises, $\rho = 0.025$. (For numerical long run implications, shown in Online-Appendix A.4, we compare results for $\rho = 0.02$ vis-à-vis $\rho = 0.03$.)

The capital income tax rate is reported in Piketty and Zucman (2014b). It slightly fluctuates over time. Considering average tax rates between 1970-2010, for Germany this gives us $\tau_r = 0.18$, for France $\tau_r = 0.19$, and for the US $\tau_r = 0.22$. The capital tax rate for the UK is not available in Piketty and Zucman (2014b). According to the OECD tax database (2015), the net top statutory dividend tax rate "to be paid at the shareholder level, taking account of all types of reliefs and gross-up provisions at the shareholder level" is similar in the US and the UK. Capital tax rates may thus be viewed as similar in the four considered countries and around 0.2. As we cannot predict future changes in tax rates, we employ $\tau_r = 0.2$ for all displayed transitional dynamics.

According to Propositions 2-6, the amount of land per house, ψ, does not affect the factor allocation, factor prices, wealth-to-income ratios, income shares and investment rates in the long run. We set the arbitrary value $\psi = 1$. Similarly, the cost parameter ξ (residential land development costs) does not have an impact on the long-run equilibrium. It does, however, affect the speed of N-dynamics along the transition. We set $\xi = 100$.

Finally, we need to specify α. We relate to the evidence by Rognlie (2015, Fig. 6), who decomposes the net capital share of corporate sector value-added in the US into the return on equipment, structures, land, and pure profits (time series since 1950). He reports that pure profits fluctuate around zero, whereas the land income share is fluctuating around 0.04. The US corporate sector incorporates the housing sector. Thus, the land income share, LIS, is the average of the land income share in the non-residential sector, $R^Z Z^Y/Y = 1-\alpha-\beta$ (according to (65)), and the land income share in the housing sector, $\Im = R^Z Z^N/H$. Consequently, α is given by

$$(1-\alpha-\beta)\left(1-\frac{H}{GDP}\right)+\Im\frac{H}{GDP}=LIS. \tag{147}$$

According to (56), the long run land income share in the housing sector is, for $\gamma = 0.9$, $\eta = 0.38$, $\delta^X = 0.015$ and $r^* = 0.031$ (implied by $\rho = 0.025$, and $\tau_r = 0.2$), given by 8.5 percent. Using $\Im = 0.085$ together with $\beta = 0.69$ and $\frac{H}{GDP} = 0.12$ in (147) suggests $\alpha = 0.28$ for $LIS = 0.04$.

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Online-Appendix A: Long Term Housing & Macro Model

In Online-Appendix A.1, we invoke Walras' law as consistency check of the analytical derivation of the long run equilibrium. Online-Appendix A.2 summarizes the dynamic system, whereas Online-Appendix A.3 summarizes long run equilibrium values. These two sets of information form the basis for the numerical implementation. Online-Appendix A.4 discusses steady state implications for the various wealth-to-income ratios in the model. Online-Appendix A.5 shows that the evolution of the housing wealth-to-income ratio is rather insensitive to the amount of available land. Online-Appendix A.6 displays transitional dynamics of the non-residential wealth-to-income ratio for the calibrated model.

A.1 Consistency Check Using Walras' Law

We show that the market for the numeraire good clears in long run equilibrium, i.e., $Y = C + \delta^K K + M$ holds in steady state (market for numeraire good clears, according to equilibrium condition 10 in Definition 1, and $\dot{K} = \dot{N} = \dot{X} = 0$). Recall the following definitions hold.

$$\varrho = \frac{\beta}{1-\alpha} - \frac{\gamma(1-\eta)}{1-\gamma\eta}, \tag{148}$$

$$\mu = \frac{\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{r+\delta^X} + \frac{r+(1-\alpha)\delta^K}{r+\delta^K}\frac{(1-\eta)\gamma\delta^X}{\beta\left(r+\delta^X\right)}}{\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{r+\delta^X} + \frac{1-\gamma}{1-\alpha-\beta}\frac{r+(1-\alpha)\delta^K}{r+\delta^K}}, \tag{149}$$

$$\upsilon = \frac{(1-\alpha-\beta)\,(1-\eta)\gamma\delta^X}{\beta(1-\gamma)(r+\delta^X)}, \tag{150}$$

$$\chi = \frac{\eta^\eta\left(\frac{\beta}{1-\eta}\right)^{\frac{\gamma\eta(1-\eta)}{1-\gamma\eta}}\left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha\eta}{1-\alpha}}\left(\frac{\upsilon}{\delta^X}\right)^{\frac{1-\eta}{1-\gamma\eta}}(1-\mu)\left(\frac{(1-\alpha-\beta)\gamma}{(1-\gamma)(r+\delta^X)}\right)^{\frac{\eta(1-\gamma)}{1-\gamma\eta}}}{(1-\upsilon)\mu^{1-\eta+\frac{\beta\eta}{1-\alpha}}}, \tag{151}$$

$$D = \frac{L}{Z}. \tag{152}$$

In steady state, the following relationships hold (we suppress superscript (*)):

$$K = \frac{\alpha}{\beta}\frac{w}{r+\delta^K}L^Y, \tag{153}$$

$$\frac{K}{Y} = \frac{\alpha}{r+\delta^K}, \tag{154}$$

$$w = \beta\left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha}{1-\alpha}}\left(B^Y\right)^{\frac{1}{1-\alpha}}\left(\frac{\mu}{D}\right)^{\frac{1-\alpha-\beta}{1-\alpha}}, \tag{155}$$

$$C = \frac{pS}{\theta}, \tag{156}$$

$$\frac{L^X}{L} = \frac{\upsilon(1-\mu)}{(1-\upsilon)\mu} = 1-\frac{L^Y}{L} \tag{157}$$

$$\Longrightarrow \frac{L^Y}{L} = \frac{1}{1-\upsilon}-\frac{\upsilon}{1-\upsilon}\frac{1}{\mu}, \tag{158}$$

$$\frac{S}{L} = \chi^\gamma\left(\frac{1-\mu}{\psi(1-\upsilon)}\right)^{1-\gamma}\frac{B^h\left(B^X\right)^\gamma\left(B^Y\right)^{\frac{\gamma\eta}{1-\alpha}}}{D^{1-\gamma+\frac{(1-\alpha-\beta)\gamma\eta}{1-\alpha}}}, \tag{159}$$

$$B^h p = \left(\frac{D}{\mu}\right)^{\varrho(1-\gamma\eta)}\frac{\left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha(1-\gamma\eta)}{1-\alpha}}\left(B^Y\right)^{\frac{1-\gamma\eta}{1-\alpha}}\left(\frac{\psi(1-\alpha-\beta)}{1-\gamma}\right)^{1-\gamma}}{\left(\eta B^M\right)^{\gamma\eta}\left(\frac{B^X\gamma}{r+\delta^X}\right)^{\gamma}\left(\frac{1-\eta}{\beta}\right)^{\gamma(1-\eta)}}, \tag{160}$$

$$\frac{N}{Z} = \frac{1-\mu}{\psi(1-\nu)}, \tag{161}$$

$$M = \left(L^X\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}}\left(\frac{\gamma\eta p B^h\left(\delta^X N\right)^{1-\gamma}\left(B^X\right)^\gamma}{r+\delta^X}\right)^{\frac{1}{1-\gamma\eta}}. \tag{162}$$

We first derive

$$
\begin{aligned}
\frac{Y-C-\delta^K K}{K} &= \frac{Y}{K}-\frac{C}{K}-\delta^K \\
&= \frac{r+\delta^K}{\alpha}-\frac{\frac{pS}{\theta}}{\frac{1}{\beta}\frac{\alpha}{r+\delta^K}\beta\left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha}{1-\alpha}}\left(B^Y\right)^{\frac{1}{1-\alpha}}\left(\frac{\mu}{D}\right)^{\frac{1-\alpha-\beta}{1-\alpha}}L^Y}-\delta^K \\
&= -\frac{\chi^\gamma\left(\frac{1-\mu}{1-\upsilon}\right)^{1-\gamma}\left(\frac{D}{\mu}\right)^{\varrho(1-\gamma\eta)}\left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha(1-\gamma\eta)}{1-\alpha}}\left(\frac{1-\alpha-\beta}{1-\gamma}\right)^{1-\gamma}\left(\frac{\beta}{1-\eta}\right)^{\gamma(1-\eta)}}{\theta D^{1-\gamma+\frac{(1-\alpha-\beta)\gamma\eta}{1-\alpha}}\left(\frac{\alpha}{r+\delta^K}\right)^{\frac{1}{1-\alpha}}\left(\frac{\mu}{D}\right)^{\frac{1-\alpha-\beta}{1-\alpha}}\left(\frac{1}{1-\upsilon}-\frac{\upsilon}{1-\upsilon}\frac{1}{\mu}\right)\eta^{\gamma\eta}\left(\frac{\gamma}{r+\delta^X}\right)^\gamma}+ \\
&\quad \frac{r+(1-\alpha)\delta^K}{\alpha} \\
&= -\frac{\left(\eta^\eta\left(\frac{\upsilon}{\delta^X}\right)^{\frac{1-\eta}{1-\gamma\eta}}\left(\frac{1-\alpha-\beta}{1-\gamma}\frac{\gamma}{r+\delta^X}\right)^{\frac{\eta(1-\gamma)}{1-\gamma\eta}}\left(\frac{\beta}{1-\eta}\right)^{\frac{\gamma\eta(1-\eta)}{1-\gamma\eta}}\right)^\gamma\frac{1-\mu}{1-\upsilon}}{\mu^{\left(1-\eta+\frac{\beta\eta}{1-\alpha}\right)\gamma}\theta D^{1-\gamma+\frac{(1-\alpha-\beta)\gamma\eta}{1-\alpha}}\frac{\alpha}{r+\delta^K}\left(\frac{1}{1-\upsilon}-\frac{\upsilon}{1-\upsilon}\frac{1}{\mu}\right)\eta^{\gamma\eta}\left(\frac{\gamma}{r+\delta^X}\right)^\gamma}\times \\
&\quad \left(\frac{D}{\mu}\right)^{1-\frac{\beta\gamma\eta}{1-\alpha}-\gamma(1-\eta)}\left(\frac{1-\alpha-\beta}{1-\gamma}\right)^{1-\gamma}\left(\frac{\beta}{1-\eta}\right)^{\gamma(1-\eta)}+ \\
&\quad \frac{r+(1-\alpha)\delta^K}{\alpha}. \qquad (163)
\end{aligned}
$$

Rearranging further, we obtain

$$
\begin{aligned}
\frac{Y-C-\delta^K K}{K} &= -\frac{\left(\frac{\upsilon}{\delta^X}\right)^{\frac{(1-\eta)\gamma}{1-\gamma\eta}}\left(\frac{D}{\mu}\right)^{1-\frac{\beta\gamma\eta}{1-\alpha}-\gamma(1-\eta)}\left(\frac{1-\alpha-\beta}{1-\gamma}\right)^{\frac{1-\gamma}{1-\gamma\eta}}\left(\frac{\beta}{1-\eta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}}}{\mu^{\left(1-\eta+\frac{\beta\eta}{1-\alpha}\right)\gamma}\theta D^{1-\gamma+\frac{(1-\alpha-\beta)\gamma\eta}{1-\alpha}}\frac{\alpha}{r+\delta^K}\left(\frac{\gamma}{r+\delta^X}\right)^{\frac{(1-\eta)\gamma}{1-\gamma\eta}}}\frac{1-\mu}{\frac{\mu-\upsilon}{\mu}}+ \\
&\quad \frac{r+(1-\alpha)\delta^K}{\alpha} \\
&= \frac{r+(1-\alpha)\delta^K}{\alpha}-\frac{r+\delta^K}{\alpha}\frac{1}{\theta}\left(\frac{1-\alpha-\beta}{1-\gamma}\right)^{\frac{1-\gamma}{1-\gamma\eta}}\times \\
&\quad \left(\frac{\beta\upsilon}{1-\eta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}}\left(\frac{r+\delta^X}{\gamma\delta^X}\right)^{\frac{(1-\eta)\gamma}{1-\gamma\eta}}\frac{1-\mu}{\mu-\upsilon} \qquad (164)
\end{aligned}
$$

For $Y = C + \delta^K K + M$ to hold, this must be equal to

$$\begin{aligned}
\frac{M}{K} &= \frac{\left(L^X\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{\gamma\eta\left(\frac{D}{\mu}\right)^{\varrho(1-\gamma\eta)} \left(B^Y\right)^{\frac{1-\gamma\eta}{1-\alpha}} \left(\frac{\psi(1-\alpha-\beta)}{1-\gamma}\right)^{1-\gamma} \left(\delta^X N\right)^{1-\gamma} \left(B^X\right)^{\gamma}}{(r+\delta^X)^{1-\gamma}\eta^{\gamma\eta}(B^X\gamma)^{\gamma}\left(\frac{1-\eta}{\beta}\right)^{\gamma(1-\eta)}} \right)^{\frac{1}{1-\gamma\eta}}}{\frac{\alpha}{r+\delta^K}\left(B^Y\right)^{\frac{1}{1-\alpha}} \left(\frac{\mu}{D}\right)^{\frac{1-\alpha-\beta}{1-\alpha}} L^Y} \\
&= \frac{\left(\frac{\upsilon}{1-\upsilon}\left[\frac{1}{\mu}-1\right]\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \eta \left(\frac{\left(B^Y\right)^{\frac{1-\gamma\eta}{1-\alpha}} (1-\alpha-\beta)^{1-\gamma} \left(\delta^X \frac{1-\mu}{1-\upsilon}\right)^{1-\gamma}}{(r+\delta^X)^{1-\gamma}\left(\frac{1-\eta}{\beta}\right)^{\gamma(1-\eta)}} \right)^{\frac{1}{1-\gamma\eta}} \left(\frac{\gamma}{1-\gamma}\right)^{\frac{1-\gamma}{1-\gamma\eta}}}{\frac{\alpha}{r+\delta^K}\left(B^Y\right)^{\frac{1}{1-\alpha}} \left(\frac{1}{1-\upsilon} - \frac{\upsilon}{1-\upsilon}\frac{1}{\mu}\right) \mu^{\frac{1-\gamma}{1-\gamma\eta}}} \\
&= \left(\frac{\gamma\left(1-\alpha-\beta\right)\delta^X}{(1-\gamma)(r+\delta^X)}\right)^{\frac{1-\gamma}{1-\gamma\eta}} \left(\frac{\beta\upsilon}{1-\eta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \frac{1-\mu}{\mu-\upsilon}\frac{r+\delta^K}{\alpha}\eta. \qquad (165)
\end{aligned}$$

Recall that μ is implicitly defined by

$$\frac{1-\alpha-\beta}{1-\gamma}\left(\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{r+\delta^X}\right)\frac{1-\mu}{\mu-\upsilon} = \frac{r+(1-\alpha)\delta^K}{r+\delta^K}. \qquad (166)$$

The market for the numeraire good thus clears if

$$\begin{aligned}
& \frac{r+(1-\alpha)\delta^K}{r+\delta^K} \\
= \; & \left(\frac{\gamma\left(1-\alpha-\beta\right)\delta^X}{(1-\gamma)(r+\delta^X)}\right)^{\frac{1-\gamma}{1-\gamma\eta}} \left(\frac{\beta\upsilon}{1-\eta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \frac{1-\mu}{\mu-\upsilon}\eta + \\
& \frac{1}{\theta}\left(\frac{1-\alpha-\beta}{1-\gamma}\right)^{\frac{1-\gamma}{1-\gamma\eta}} \left(\frac{\beta\upsilon}{1-\eta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{r+\delta^X}{\gamma\delta^X}\right)^{\frac{(1-\eta)\gamma}{1-\gamma\eta}} \frac{1-\mu}{\mu-\upsilon} \\
\Longleftrightarrow \; & \left(\frac{1-\alpha-\beta}{1-\gamma}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} = \left(\frac{\beta\upsilon}{1-\eta}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{r+\delta^X}{\gamma\delta^X}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \\
\Longleftrightarrow \; & \left(\frac{1-\alpha-\beta}{1-\gamma}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} = \left(\frac{(1-\alpha-\beta)\,\gamma\delta^X}{(1-\gamma)(r+\delta^X)}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}} \left(\frac{r+\delta^X}{\gamma\delta^X}\right)^{\frac{\gamma(1-\eta)}{1-\gamma\eta}}, \qquad (167)
\end{aligned}$$

which holds. ■

A.2 Dynamic System

The Long Term Housing & Macro Model is fully described by seven differential equations plus a set of static equations. The differential equations read as:

$$\dot{X} = B^X M^\eta \left(L^X\right)^{1-\eta} - \delta^X X, \tag{168}$$

$$\dot{N} = \frac{q^N - \psi P^Z}{\xi}, \tag{169}$$

$$\dot{W} = rW + wL - C - pS, \tag{170}$$

$$\dot{C} = C\left[(1-\tau_r)r - \rho\right], \tag{171}$$

$$\dot{q}^N = rq^N - \pi, \tag{172}$$

$$\dot{q}^X = \left(r + \delta^X\right) q^X - R^X, \tag{173}$$

$$\dot{P}^Z = rP^Z - R^Z, \tag{174}$$

where K_0, N_0, X_0 are given. The set of static equations is given by:

$$Z^Y + \psi N = Z, \quad L^X + L^Y = L, \quad W = K + q^N N + q^X X + P^Z Z^Y, \tag{175}$$

$$x = \frac{X}{N}, \quad h = B^h x^\gamma, \quad S = Nh, \tag{176}$$

$$w = (1-\eta)\eta^{\frac{\eta}{1-\eta}} \left(B^X q^X\right)^{\frac{1}{1-\eta}}, \quad R^Z = B^Y (1-\alpha-\beta) \left(\frac{\alpha}{\beta}\frac{w}{r+\delta^K}\right)^\alpha \left(\frac{L^Y}{Z^Y}\right)^{\alpha+\beta}, \tag{177}$$

$$R^X = pB^h \gamma x^{\gamma-1}, \quad \pi = (1-\gamma)pB^h x^\gamma, \tag{178}$$

$$\theta C = pS, \quad \frac{K}{L^Y} = \frac{\alpha}{\beta}\frac{w}{r+\delta^K}, \quad \frac{L^X}{M} = \left(\frac{1}{q^X \eta B^X}\right)^{\frac{1}{1-\eta}}, \tag{179}$$

$$\left(\frac{r+\delta^K}{\alpha}\right)^\alpha \left(\frac{w}{\beta}\right)^{1-\alpha} = B^Y \left(\frac{Z^Y}{L^Y}\right)^{1-\alpha-\beta}. \tag{180}$$

In total, there are 21 equations and 21 endogenous variables: X, x, N, W, C, π, q^N, P^Z, q^X, K, Z^Y, L^X, L^Y, M, S, h, r, w, R^X, R^Z, p.

A.3 Steady State Values

According to Section 7.3, we can summarize long run values as follows.

$$r^* = \frac{\rho}{1-\tau_r}. \tag{181}$$

$$Z^{Y*} = Z\frac{\mu(r^*) - \upsilon(r^*)}{1-\upsilon(r^*)}, \quad N^* = \frac{Z - Z^{Y*}}{\psi}, \tag{182}$$

$$L^{X*} = L\frac{\upsilon(r^*)\left[1-\mu(r^*)\right]}{\left[1-\upsilon(r^*)\right]\mu(r^*)}, \quad L^{Y*} = L - L^{X*}, \tag{183}$$

$$w^* = \beta\left(\frac{\alpha}{r^*+\delta^K}\right)^{\frac{\alpha}{1-\alpha}}(B^Y)^{\frac{1}{1-\alpha}}\left(\frac{D}{\mu(r^*)}\right)^{-\frac{1-\alpha-\beta}{1-\alpha}}, \tag{184}$$

$$R^{Z*} = (1-\alpha-\beta)\left(\frac{\alpha}{r^*+\delta^K}\right)^{\frac{\alpha}{1-\alpha}}(B^Y)^{\frac{1}{1-\alpha}}\left(\frac{D}{\mu(r^*)}\right)^{\frac{\beta}{1-\alpha}}, \tag{185}$$

$$P^{Z*} = \frac{R^{Z*}}{r^*}, \quad q^{N*} = \frac{\psi R^{Z*}}{r^*}, \tag{186}$$

$$p^* = \left(\frac{D}{\mu(r^*)}\right)^{\varrho(1-\gamma\eta)}\frac{\left(\frac{\alpha}{r^*+\delta^K}\right)^{\frac{\alpha(1-\gamma\eta)}{1-\alpha}}\left(B^Y\right)^{\frac{1-\gamma\eta}{1-\alpha}}\left(\frac{\psi(1-\alpha-\beta)}{1-\gamma}\right)^{1-\gamma}}{B^h\eta^{\gamma\eta}\left(\frac{B^X\gamma}{r^*+\delta^X}\right)^{\gamma}\left(\frac{1-\eta}{\beta}\right)^{\gamma(1-\eta)}}, \tag{187}$$

$$X^* = Z\chi(r^*)D^{1-\eta+\frac{\beta\eta}{1-\alpha}}B^X\left(B^Y\right)^{\frac{\eta}{1-\alpha}}, \tag{188}$$

$$x^* = \frac{X^*}{N^*}, \quad h^* = B^h(x^*)^\gamma, \quad S^* = N^*h^*,$$

$$C^* = \frac{p^*S^*}{\theta}, \tag{189}$$

$$\pi^* = \psi R^{Z*}, \tag{190}$$

$$q^{X*} = \frac{\gamma p^* S^*}{(r^*+\delta^X)X^*}, \quad R^{X*} = (r^*+\delta^X)q^X, \tag{191}$$

$$M^* = \left(\eta B^X q^{X*}\right)^{\frac{1}{1-\eta}}L^{X*}, \tag{192}$$

$$K^* = \frac{\alpha}{\beta}\frac{w^*L^{Y*}}{r^*+\delta^K}, \tag{193}$$

$$W^* = K^* + q^{N*}N^* + q^{X*}X^* + P^{Z*}Z^{Y*}, \tag{194}$$

where we used definitions

$$\varrho = \frac{\beta}{1-\alpha} - \frac{\gamma(1-\eta)}{1-\gamma\eta} \overset{(A1)}{>} 0, \tag{195}$$

$$\upsilon(r) = \frac{(1-\alpha-\beta)(1-\eta)\gamma\delta^X}{\beta(1-\gamma)(r+\delta^X)}, \tag{196}$$

$$\mu(r) = \frac{\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{r+\delta^X} + \frac{r+(1-\alpha)\delta^K}{r+\delta^K}\frac{(1-\eta)\gamma\delta^X}{\beta\left(r+\delta^X\right)}}{\frac{1}{\theta} + \frac{\eta\gamma\delta^X}{r+\delta^X} + \frac{1-\gamma}{1-\alpha-\beta}\frac{r+(1-\alpha)\delta^K}{r+\delta^K}}, \tag{197}$$

$$\chi(r) = \frac{\eta^\eta \left(\frac{\beta}{1-\eta}\right)^{\frac{\gamma\eta(1-\eta)}{1-\gamma\eta}} \left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha\eta}{1-\alpha}} \left(\frac{\nu}{\delta^X}\right)^{\frac{1-\eta}{1-\gamma\eta}} [1-\mu(r)] \left(\frac{1-\alpha-\beta}{1-\gamma}\frac{\gamma}{r+\delta^X}\right)^{\frac{\eta(1-\gamma)}{1-\gamma\eta}}}{[1-\upsilon(r)]\mu(r)^{1-\eta+\frac{\beta\eta}{1-\alpha}}}. \tag{198}$$

A.4 Long-Run Implications for Wealth-to-Income Ratios

Table A.1 shows (in percent) long run implications for wealth-to-NDP ratios as resulting from the calibrated Long Term Housing & Macro Model, assuming alternative subjective discount rates, ρ, and capital income tax rates, τ_r.

The annual average of the housing wealth-to-NDP ratio, $\mathfrak{H}^{NDP}$, in the 2000s was 217 percent in the US with a peak of 254 percent in 2006 before the financial crisis.[48] In the UK, Germany and France, $\mathfrak{H}^{NDP}$ was 271 percent, 217 percent and 285 percent in the 2000s, respectively. The calibrated model under $\tau_r = 0.2$ implies the long run value, $\mathfrak{H}^{NDP*}$, to be 357 percent for $\rho = 0.03$ and 478 percent for $\rho = 0.02$.[49] Our analysis thus suggests that, in the longer run, the US housing capital will rise considerably above the pre-crisis level if the capital income tax rate remains similar.

Maintaining $\tau_r = 0.2$, the implied non-residential wealth-to-NDP ratio is $\mathfrak{N}^{NDP*} =$ 349 percent for $\rho = 0.03$ and to $\mathfrak{N}^{NDP*} = 440$ percent for $\rho = 0.02$. About three quarters are attributed to physical capital and one quarter to non-residential land wealth. In the 2000s, $\mathfrak{N}^{NDP}$ was 249 percent in the US, 241 percent in the UK, 139 percent in Germany, and 190 percent in France. The implied future increase from current levels

[48] With respect to stylized facts on wealth-to-income ratios, we again refer to the data provided by Piketty and Zucman (2014b).

[49] Recall that $\mathfrak{H}^{NDP*} = 4.1$ for $\rho = 0.025$ (Figure 1).

partly reflects the growing importance of land scarcity in the process of economic development. However, as discussed in more detail in Online-Appendix A.6, the fact that we do not distinguish rural and urban land may contribute to an unrealistically high long run non-residential land wealth-to-NDP ratio, $\mathfrak{Z}^{NDP*}$. Without further amendments, if the current tax system remains in place, the implied long run wealth-to-NDP ratio, $\mathfrak{W}^{NDP*}$, is in the range from 706 to 918 percent, depending on the subjective discount rate, ρ. The sensitivity of long run wealth-to-GDP ratios to ρ is rooted in the fact that the PDV of asset values, particularly land that does not depreciate, is heavily dependent on the rate at which future returns are discounted.

ρ	τ_r	$\mathfrak{H}^{NDP*}$	$\mathfrak{K}^{NDP*}$	$\mathfrak{Z}^{NDP*}$	$\mathfrak{N}^{NDP*}$	$\mathfrak{W}^{NDP*}$
0.02	0.15	498	320	136	456	954
0.03	0.15	374	274	88	361	735
0.02	0.2	478	313	127	440	918
0.03	0.2	357	267	82	349	706
0.02	0.25	457	306	119	425	882
0.03	0.25	340	259	76	335	676

Table A.1. Long run implications for wealth-to-NDP ratios.

Notes: All values are expressed in percent. Results are based on the following set of parameters: $\alpha = 0.28$, $\beta = 0.69$, $\gamma = 0.9$, $\eta = 0.38$, $\theta = 0.22$, $\delta^X = 0.015$, $\delta^K = 0.07$. Recall $\mathfrak{H}^{NDP*} = \frac{P^{H*}N^*}{NDP^*}$, $\mathfrak{K}^{NDP*} = \frac{K^*}{NDP^*}$, $\mathfrak{Z}^{NDP*} = \frac{P^{Z*}Z^{Y*}}{NDP^*}$, $\mathfrak{N}^{NDP*} = \mathfrak{K}^{NDP*} + \mathfrak{Z}^{NDP*}$, $\mathfrak{W}^{NDP*} = \frac{W^*}{NDP^*} = \mathfrak{H}^{NDP*} + \mathfrak{N}^{NDP*}$.

Table A.1 also displays the sensitivity of wealth-to-income ratios with respect to capital income taxation. We start with lowering τ_r to 15 percent. As the long run interest rate, r^*, is slightly reduced, the implied wealth-to-NDP ratios increase somewhat. The implied housing wealth-to-NDP ratio, $\mathfrak{H}^{NDP*}$, now becomes almost 500 percent for $\rho = 0.02$ and associated long run wealth-to-NDP ratio, $\mathfrak{W}^{NDP*}$, is 954 percent without amending non-residential land wealth. These values may be considered as upper bounds.

Raising τ_r to 25 percent, $\rho = 0.03$ implies that $\mathfrak{H}^{NDP*}$ is 340 percent, which may be considered as lower bound.

According to Table A.2, the implied long-run labor share in income, $\mathfrak{L}^*$, is about 60 percent. It does neither critically depend on ρ nor on τ_r. Comparing it for the latest available year, the labor income share was 57.2 percent in the year 2012 in the US and 62.1 percent in 2011 in the UK (Karabarbounis and Neiman, 2014, "CLS KN merged"). For $\tau_r = 0.2$, the implied long run employment fraction in residential construction, l^{X*}, is 4.9 and 3.9 percent for ρ equal to three and two percent, respectively. It is slightly decreasing in τ_r and somewhat higher than the recent US value of 2.7 percent used for calibrating the model. The result is associated with a higher stock of structures, X, in the long run compared to the current (off steady state) one, in line with our prediction of a rising $\mathfrak{H}$. A high construction labor employment share copes with the constant depreciation rate of structures, δ^X, given a high value of X. The implied long run fraction of land devoted to the housing sector in the economy, $\mathfrak{z}^{N*}$, is about 36-37 percent.

ρ	τ_r	$\mathfrak{L}^*$	l^{X*}	$\mathfrak{z}^{N*}$	$\mathfrak{s}^*$	$\mathfrak{s}^{K*}$	$\mathfrak{s}^{H*}$
0.02	0.15	60.3	5.1	36.0	22.3	17.4	4.9
0.03	0.15	59.7	4.1	36.9	19.4	15.5	3.9
0.02	0.2	60.2	4.9	36.1	21.9	17.1	4.8
0.03	0.2	59.6	3.9	37.0	18.9	15.1	3.8
0.02	0.25	60.1	4.7	36.3	21.4	16.8	4.6
0.03	0.25	59.5	3.8	37.1	18.4	14.8	3.6

Table A.2. Long run implications for labor income share, net capital income shares, allocation variables, and investment rates.

Notes: All values are expressed in percent. Results are based on the following set of parameters: $\alpha = 0.28$, $\beta = 0.69$, $\gamma = 0.9$, $\eta = 0.38$, $\theta = 0.22$, $\delta^X = 0.015$, $\delta^K = 0.07$. Recall $\mathfrak{L}^* = \frac{w^* L}{GDP^*}$, $l^{X*} = \frac{L^{X*}}{L}$, $\mathfrak{z}^{N*} = \frac{Z^{N*}}{Z}$, $\mathfrak{s}^* = \frac{\mathcal{I}^*}{GDP^*}$, $\mathfrak{s}^{K*} = \frac{I^{K*}}{GDP^*}$, $\mathfrak{s}^{H*} = \frac{I^{N*} + q^{X*} I^{X*}}{GDP^*}$.

For $\tau_r = 0.2$, the long run investment rate for the baseline calibration, $\mathfrak{s}^*$, is 18.9 and 21.9 percent for ρ equal to three and two percent, respectively, and may be compared to the US gross private domestic investment rate of 16.5 percent for the year 2014 (Bureau of Economic Analysis, 2015a). For the period 1969-2014, the average ratio of US non-residential investment to GDP, a measure for $\mathfrak{s}^K$, was 12.6 percent, which approximately also was the average value of the 1970s, 1980s, 1990s and 2000s (Bureau of Economic Analysis, 2015c, Tab. 1.1.10). The value is somewhat lower than the long run value for $\mathfrak{s}^{K*}$ that is implied by our model. The implied long run residential investment rate, $\mathfrak{s}^{H*} = \mathfrak{s}^* - \mathfrak{s}^{K*}$, is in the range of 3.6 to 4.9 percent.

A.5 Evolution of Housing Wealth-to-Income under Expansion of Land

Figure A.1 shows the evolution of housing wealth (relative to income) under two different scenarios. The solid line shows the case of a fixed land endowment, i.e., $Z = const.$ The dashed line assumes that Z_t increases over time by a factor of 2. This captures the case where the amount of economically usable land increases over time. The difference for the time path of the housing wealth-to-income ratio is minor. The reason is that there are opposite quantity and price effects. Under scenario "$Z = const.$" the number of housing projects N changes only slightly, but the house price P_t^H increases substantially. Under scenario "Z_t increases", the number of housing projects rises substantially, but the house price P_t^H increases by less compared to scenario "$Z = const.$".

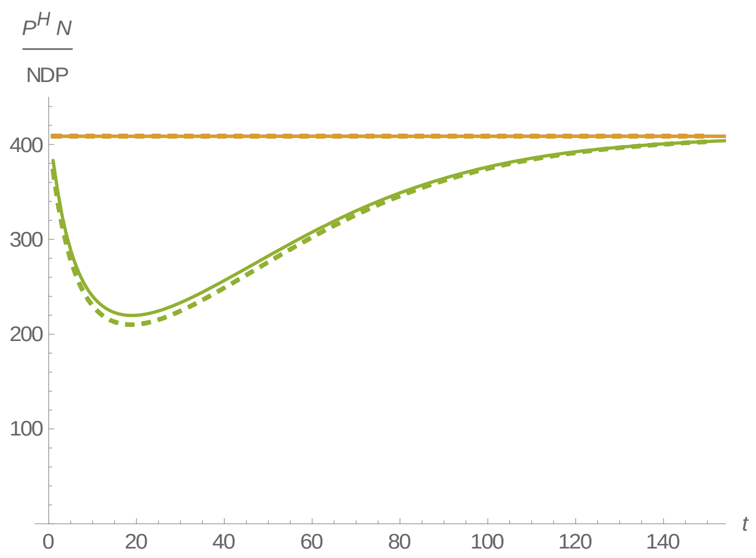

Figure A.1. Housing wealth to income (NDP) over 140 years under alternative assumptions on land endowment.

Notes. The economy starts in a steady state. The solid line shows the evolution of $\mathfrak{H}_t$ in the case of a fixed land endowment; the dashed line shows the case where the amount of economically usable land increase over time, i.e., Z_t increases by a factor of 2. L_t, B_t^Y and B_t^X increase by a factor of 2 according to logistic functions. Other parameters: $\alpha = 0.28$, $\beta = 0.69$, $\gamma = 0.9$, $\eta = 0.38$, $\theta = 0.22$, $\delta^X = 0.015$, $\delta^K = 0.07$, $B^h = 1$, $\rho = 0.025$, $\tau_r = 0.2$, $\psi = 1$, $\xi = 100$.

A.6 Non-Residential Wealth

We finally report the implications of the experiment sketched in Section 4.2.1 for non-residential wealth.

Figure A.2 displays the evolution of non-residential wealth relative to NDP, $\mathfrak{N}_t^{NDP}$, for the four countries under consideration. The dotted (red / purple) lines show the empirical data, whereas the solid (blue) lines display the model-based time paths. The match is less close compared to the housing wealth-to-income ratio (Figure 1). In particular, for Germany the model implies a strong increase, whereas the data show a moderate U-shaped development. Also for France the model implies a somewhat faster increase in non-residential wealth, relative to income, compared to empirical data. This is due to two reasons: *First*, Germany starts with the lowest value of physical capital (relative to the steady state), which again appears reasonable with respect to war destructions during World War II ($\mathfrak{N}_{1955}^{NDP} \approx 105$ percent). The model economy builds up the capital stock quite rapidly. This implication would be different, if one assumed (convex) adjustment costs for physical capital $-$ a feature that we left out for simplicity, reflecting our focus on housing capital. The same feature would improve the picture for France that starts with only a slightly higher value than Germany ($\mathfrak{N}_{1955}^{NDP} \approx 127$ percent). In 2005 the empirical value is about 190 percent, while the model displays a value of about 255 percent. The match is much better for the U.K. and the U.S. *Second*, the Long Term Housing & Macro Model rests on one simplifying assumption that is important when it comes to non-residential wealth. It does not distinguish between urban and rural land. That is, there is one single region that hosts both production and housing. This region should in fact be interpreted as predominantly urban, given that about 70 to 80 percent of the population in advanced countries lives in cities. Therefore, the Long Term Housing & Macro Model values the entire non-residential land, which comprises non-residential urban and non-residential rural land in reality, at the urban land price.[50] Given the large urbanization rates in advanced countries, this land valuation bias affects housing wealth to a much lesser extent than non-residential wealth. As a result, the non-residential wealth-to-income ratio, $\mathfrak{N}_t^{NDP}$, is overestimated due to an overvaluation of (non-residential) land, $P_t^Z Z_t^Y$.[51]

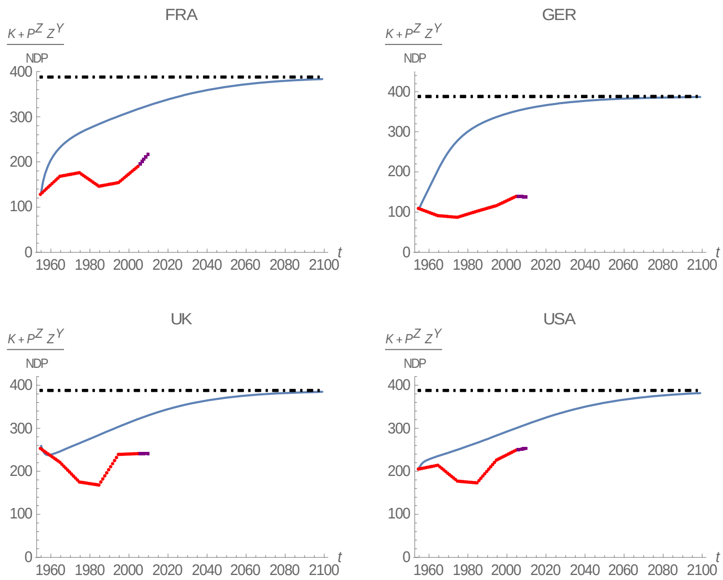

Figure A.2. Non-residential wealth (relative to NDP) from 1955 until 2100.

[50]The same criticism applies, of course, to any other one-regional macro model with production and housing. The issue of land heterogeneity and associated land price differentials has already been brought up in the context of macroeconomics and housing by Sachs and Boone (1988).

[51]A simple possibility to amend this valuation bias is to value non-residential land, Z_t^Y, at an adjusted land price $\bar{P}_t^Z$ that is a weighted average price of land across urban and rural regions (assuming a price wedge between the two regions). It can be represented as $\bar{P}_t^Z = \kappa P_t^Z$ with $0 < \kappa < 1$.

Notes. See Figure 1.

Additional Reference

Sachs, Jeffrey and Peter Boone (1988). Japanese Structural Adjustment and the Balance of Payments, *Journal of the Japanese and International Economics* 2, 286-327.

Online-Appendix B: Canonical Housing & Macro Model

In Online-Appendix B.1, we characterize analytically the long run equilibrium of the canonical model. In Online-Appendix B.2, we invoke Walras' law as a consistency check of the analytical derivation of the long run equilibrium. In Online-Appendix B.3, we summarize the dynamical system of the canonical model.

B.1 Long Run Equilibrium

Define $l^X \equiv L^X/L$, $l^Y \equiv L^Y/L$, $k^X \equiv K^X/L$, $k^Y \equiv K^Y/L$, $k \equiv K/L$, $h \equiv H/L$ and $\bar{z} \equiv \bar{Z}/L$. Thus, (39) and (40) can be written as $k^X + k^Y = k$ and $l^X + l^Y = 1$, respectively.

Proposition B.1. *In the canonical model, (i) the long run allocation of labor, (l^{X*}, l^{Y*}), is independent of both new land per capita, $\bar{z}$, and productivity parameters B^X, B^Y, B^H; (ii) the long run capital-labor ratio (k^*) is independent of $\bar{z}$, B^X, B^H, and increasing in B^Y.*

Proof of Proposition B.1. Denote by p^X the price of structures. Moreover, define $k^X \equiv K^X/L$, $k^Y \equiv K^Y/L$. Profit maximization in the X and Y sector then implies

$$\alpha B^Y \left(\frac{l^Y}{k^Y}\right)^{1-\alpha} = \gamma p^X B^X \left(\frac{l^X}{k^X}\right)^{1-\gamma} = r + \delta^K, \tag{199}$$

$$(1-\alpha) B^Y \left(\frac{k^Y}{l^Y}\right)^{\alpha} = (1-\gamma) p^X B^X \left(\frac{k^X}{l^X}\right)^{\gamma} = w. \tag{200}$$

Combining (199) and (200) implies

$$k^X = \frac{\gamma}{1-\gamma} \frac{w l^X}{r + \delta^K}, \tag{201}$$

$$k^Y = \frac{\alpha}{1-\alpha} \frac{w l^Y}{r + \delta^K}. \tag{202}$$

Thus, using resource constraints $k^X + k^Y = k$ and $l^X + l^Y = 1$, we obtain

$$k = \frac{w}{r + \delta^K}\left(\frac{\gamma}{1-\gamma}l^X + \frac{\alpha}{1-\alpha}(1 - l^X)\right). \tag{203}$$

Moreover, according to (199), (200) and resource constraints, we find

$$p^X = \frac{(1-\alpha)B^Y\left(\frac{k-k^X}{1-l^X}\right)^{\alpha}}{(1-\gamma)B^X\left(\frac{k^X}{l^X}\right)^{\gamma}}, \tag{204}$$

$$w = (1-\alpha)B^Y\left(\frac{k-k^X}{1-l^X}\right)^{\alpha}. \tag{205}$$

The current-value Hamiltonian of the representative housing firm associated with its profit maximization problem together with the necessary first-order conditions can be expressed as

$$\mathcal{H}^H \equiv pH - p^X X - R^Z Z^H + q^H\left[B^H X^{\beta}\bar{Z}^{1-\beta} - \delta^H H\right], \tag{206}$$

$$\left[\frac{\partial \mathcal{H}^H}{\partial X} =\right] - p^X + \beta q^H B^H\left(\frac{\bar{Z}}{X}\right)^{1-\beta} = 0, \tag{207}$$

$$\left[\frac{\partial \mathcal{H}^H}{\partial \bar{Z}} =\right] - R^Z + (1-\beta)q^H B^H\left(\frac{X}{\bar{Z}}\right)^{\beta} = 0, \tag{208}$$

$$\left[-\frac{\partial \mathcal{H}^H}{\partial H} =\right] - p + \delta^H q^H = \dot{q}^H - rq^H. \tag{209}$$

Substituting (35) into (207) and (208), we have

$$p^X = \beta q^H\frac{\tilde{B}^H}{B^X}\left(\frac{\bar{z}}{(k^X)^{\gamma}(l^X)^{1-\gamma}}\right)^{1-\beta}, \tag{210}$$

$$R = (1-\beta)q^H\tilde{B}^H\left(\frac{(k^X)^{\gamma}(l^X)^{1-\gamma}}{\bar{z}}\right)^{\beta}, \tag{211}$$

respectively, where we used $\tilde{B}^H = B^H(B^X)^\beta$. Combining (204) and (210) leads to

$$q^H = \frac{(1-\alpha)B^Y\left(\frac{k-k^X}{1-l^X}\right)^\alpha}{(1-\gamma)\beta\tilde{B}^H\bar{z}^{1-\beta}}\frac{\left(l^X\right)^{1-\beta(1-\gamma)}}{\left(k^X\right)^{\beta\gamma}}. \tag{212}$$

Setting $\dot{H}=0$ in (36) we get

$$h = \frac{\tilde{B}^H}{\delta^H}\left(k^X\right)^{\beta\gamma}\left(l^X\right)^{\beta(1-\gamma)}\bar{z}^{1-\beta}. \tag{213}$$

Using $S=H$, analogously to (102) and (103), utility maximization of the representative household yields

$$C = \frac{pH}{\theta}, \tag{214}$$

$$\frac{\dot{C}}{C} = r-\rho. \tag{215}$$

In long run equilibrium, again, $\dot{C}=0$ implies $r^*=\rho$. Using $\dot{q}^H=0$ in (209) we find

$$p = (r+\delta^H)q^H. \tag{216}$$

Moreover, setting $\dot{A}=0$ in (38) and using (214), $S=H$, (37) and (216) leads to

$$rk + w + R^Z\bar{z} = \left(\frac{r+(1+\theta)\delta^H}{\theta}\right)q^H h. \tag{217}$$

Substituting (203), (205), (211), (212) and (213) into (217), we get

$$l^X = \frac{\left[r+(1-\alpha)\delta^K\right]\beta\theta\delta^H(1-\gamma)}{(1-\alpha)(r+\delta^K)\left[r+(1+\beta\theta)\delta^H\right]+\beta\theta\delta^H r\left(\alpha-\gamma\right)} \equiv l^{X*}. \tag{218}$$

This confirms part (i) of Proposition B.1.

Substituting (205) into (203) leads to

$$k = \frac{(1-\alpha)B^Y}{r+\delta^K}\left(\frac{\gamma}{1-\gamma}l^X + \frac{\alpha}{1-\alpha}(1-l^X)\right)\left(\frac{k-k^X}{1-l^X}\right)^\alpha, \tag{219}$$

Substituting $w = (1-\alpha)B^Y \left(k^Y/l^Y\right)^\alpha$ from (200) into (202) and using $l^X = l^{X*}$ we obtain

$$k^Y = k - k^X = \left(\frac{\alpha B^Y}{r+\delta^K}\right)^{\frac{1}{1-\alpha}} (1-l^X) \equiv k^{Y*}. \tag{220}$$

Substituting (220) into (219) and using $l^X = l^{X*}$ yields

$$k = \left(\frac{\alpha B^Y}{r+\delta^K}\right)^{\frac{1}{1-\alpha}} \left(\frac{(\gamma-\alpha)l^{X*}}{\alpha(1-\gamma)} + 1\right) \equiv k^*. \tag{221}$$

Thus, in view of (218), also part (ii) of Proposition B.1 is confirmed. ■

Proposition B.2. *In the canonical model, (i) the long run house price, q^{H*}, is decreasing in new land per capita, $\bar{z}$, increasing in B^Y, and decreasing in B^X and B^H; (ii) the long run stock of houses per capita, h^*, is increasing in B^X, B^H, B^Y, and $\bar{z}$.*

Proof of Proposition B.2. Substituting (205) into (201), we obtain

$$k^X = \frac{\gamma}{1-\gamma} \frac{(1-\alpha)B^Y l^X}{r+\delta^K} \left(\frac{k-k^X}{1-l^X}\right)^\alpha. \tag{222}$$

Next, substitute (222) into (212) and (213) and use both (220) and $l^X = l^{X*}$ to find

$$q^H = \frac{(1-\alpha)^{1-\beta\gamma}\left(B^Y\right)^{\frac{1-\beta\gamma}{1-\alpha}} \alpha^{\frac{\alpha(1-\beta\gamma)}{1-\alpha}}}{\gamma^{\beta\gamma}(1-\gamma)^{1-\beta\gamma}\beta\tilde{B}^H} \left(\frac{1}{r+\delta^K}\right)^{\frac{\alpha-\beta\gamma}{1-\alpha}} \left(\frac{l^{X*}}{\bar{z}}\right)^{1-\beta} \equiv q^{H*}, \tag{223}$$

$$h = \frac{\tilde{B}^H}{\delta^H}\left(\frac{\gamma}{1-\gamma}\right)^{\beta\gamma} (1-\alpha)^{\beta\gamma}\alpha^{\frac{\alpha\beta\gamma}{1-\alpha}} \left(\frac{B^Y}{r+\delta^K}\right)^{\frac{\beta\gamma}{1-\alpha}} \left(l^{X*}\right)^\beta \bar{z}^{1-\beta} \equiv h^*, \tag{224}$$

respectively. Parts (i) and (ii) are confirmed by (223) and (224), respectively. ■

Let housing-wealth-to-GDP ratio be defined by

$$\mathfrak{H} \equiv \frac{q^H H}{Y + pH + q^H I^H}. \tag{225}$$

Proposition B.3. *In the canonical model, a change in B^X, B^H, B^Y, and $\bar{z}$ do not* *affect (i) the long run housing-wealth-to-GDP ratio, (ii) the capital to GDP ratio, (iii) the labor share in GDP, (iv) the savings rate.*

Proof of Proposition B.3. The "house-price-to-rent ratio" in the canonical model is $\mathfrak{p}_t = q_t^H / p_t$. Using $\dot{q}^H = 0$ in (209), the long run value of the house-price-to-rent ratio is given by

$$\mathfrak{p}^* = \frac{1}{r + \delta^H}. \tag{226}$$

Using $p^*/q^{H*} = r + \delta^H$, the definition of GDP in (42), $Y/L = B^Y \left(k^Y\right)^\alpha \left(l^Y\right)^{1-\alpha}$ from (33), and long run relationship $I^H/L = \delta^H h$ in (225), we obtain

$$\mathfrak{H}^* = \frac{q^{H*}h^*}{\left(\frac{Y}{L}\right)^* + p^*h^* + q^{H*}\left(\frac{I^H}{L}\right)^*} = \frac{1}{\frac{B^Y (k^{Y*})^\alpha (l^{Y*})^{1-\alpha}}{q^{H*}h^*} + r + 2\delta^H}. \tag{227}$$

Using (223) and (224), we have

$$q^{H*}h^* = \frac{(1-\alpha)\left(B^Y\right)^{\frac{1}{1-\alpha}}}{(1-\gamma)\beta\delta^H}\left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha}{1-\alpha}} l^{X*}. \tag{228}$$

According to (220), (228) and $l^{Y*} = 1 - l^{X*}$, we obtain

$$\frac{B^Y \left(k^{Y*}\right)^\alpha \left(l^{Y*}\right)^{1-\alpha}}{q^{H*}h^*} = \frac{(1-\gamma)\beta\delta^H}{1-\alpha}\frac{1-l^{X*}}{l^{X*}}. \tag{229}$$

Using (229) in (227) and recalling (218) confirms the part (i).

Using (42), the long run capital to GDP ratio, denoted by $\mathfrak{N}^*$, can be written as

$$\mathfrak{N}^* = \frac{k^*}{\left(\frac{Y}{L}\right)^* + p^*h^* + q^{H*}\left(\frac{I^H}{L}\right)^*} = \frac{\frac{k^*}{q^{H*}h^*}}{\frac{B^Y (k^{Y*})^\alpha (l^{Y*})^{1-\alpha}}{q^{H*}h^*} + r + 2\delta^H}. \tag{230}$$

Combining (221) and (228) implies

$$\frac{k^*}{q^{H*}h^*} = \frac{\beta\delta^H}{(1-\alpha)(r+\delta^K)}\left(\gamma - \alpha + \frac{\alpha(1-\gamma)}{l^{X*}}\right). \tag{231}$$

Using (229) and (231) in (230) and recalling (218) confirms the part (ii). Using (42), the long run labor share in GDP can be written as

$$\mathfrak{L}^* = \frac{w^*}{\left(\frac{Y}{L}\right)^* + p^*h^* + q^{H*}\left(\frac{I^H}{L}\right)^*} = \frac{\frac{w^*}{q^{H*}h^*}}{\frac{B^Y(k^{Y*})^\alpha (l^{Y*})^{1-\alpha}}{q^{H*}h^*} + r + 2\delta^H}. \tag{232}$$

Using $l^Y = 1 - l^X$ in (200) and (220), we have

$$w^* = (1-\alpha)(B^Y)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{r+\delta^K}\right)^{\frac{\alpha}{1-\alpha}}. \tag{233}$$

Combining (233) and (228) implies

$$\frac{w^*}{q^{H*}h^*} = \frac{(1-\gamma)\beta\delta^H}{l^{X*}}. \tag{234}$$

Using (229) and (234) in (232) and recalling (218) confirms the part (iii).

Finally, to prove part (iv), according to (42), the long run savings rate can be rewritten as

$$\mathfrak{s}^* = 1 - \frac{\left(\frac{C}{L}\right)^* + p^*\left(\frac{S}{L}\right)^*}{\left(\frac{Y}{L}\right)^* + p^*h^* + q^{H*}\left(\frac{I^H}{L}\right)^*} = 1 - \frac{\left(\frac{1}{\theta}+1\right)(r+\delta^H)}{\frac{B^Y(k^{Y*})^\alpha (l^{Y*})^{1-\alpha}}{q^{H*}h^*} + r + 2\delta^H}, \tag{235}$$

where we divided both nominator and denominator by $q^{H*}h^*$ and used (214), $(S/L)^* = h^*$ and $p^*/q^{H*} = r + \delta$ to derive the second equation. Using (229) in (235) and recalling (218) concludes the proof. ■

B.2 Consistency Check Using Walras' Law

We show that the market for the numeraire good clears in long run equilibrium, where $\dot{K} = \dot{H} = 0$. That is, it has to hold that

$$\frac{Y^*}{L} = \frac{C^*}{L} + \delta^K \frac{K^*}{L}. \tag{236}$$

Since $Y/L = B^Y\left(k^Y\right)^\alpha \left(l^Y\right)^{1-\alpha}$, $C/L = ph/\theta$ (recall (214) and $S/L = h$) and $K/L = k$, we have to check if $B^Y \left(k^{Y*}\right)^{\alpha} \left(l^{Y*}\right)^{1-\alpha} = p^* h^*/\theta + \delta^K k^*$, i.e.,

$$\frac{B^Y \left(k^{Y*}\right)^{\alpha} \left(l^{Y*}\right)^{1-\alpha}}{q^{H*} h^*} = \frac{p^*}{\theta q^{H*}} + \frac{\delta^K k^*}{q^{H*} h^*}. \tag{237}$$

Substituting (221), (228), (229) and $p^*/q^{H*} = r + \delta^H$ into (237) it should hold that:

$$\frac{(1-\gamma)\beta\delta^H \left(1 - l^{X*}\right)}{(1-\alpha) l^{X*}} = \frac{r + \delta^H}{\theta} + \frac{\delta^K \left(\frac{\alpha B^Y}{r+\delta^K}\right)^{\frac{1}{1-\alpha}} \left(\frac{(\gamma-\alpha) l^{X*}}{\alpha(1-\gamma)} + 1\right)}{\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} (B^Y)^{\frac{1}{1-\alpha}}}{(1-\gamma)\beta\delta^H} \left(\frac{1}{r+\delta^K}\right)^{\frac{\alpha}{1-\alpha}} l^{X*}}, \tag{238}$$

being equivalent to

$$l^{X*} = \frac{\left[r + (1-\alpha)\delta^K\right](1-\gamma)\beta\theta\delta^H}{\left(r+\delta^K\right)\left[(1-\alpha)(r+\delta^H) + (1-\gamma)\beta\theta\delta^H\right] + \beta\theta\delta^H\delta^K(\gamma-\alpha)}. \tag{239}$$

It is easy to show that

$$\begin{aligned} & \left(r+\delta^K\right)\left[(1-\alpha)(r+\delta^H) + (1-\gamma)\beta\theta\delta^H\right] + \beta\theta\delta^H\delta^K(\gamma-\alpha) \\ = \; & (1-\alpha)(r+\delta^K)\left[r + (1+\beta\theta)\delta^H\right] + \beta\theta\delta^H r\left(\alpha-\gamma\right). \end{aligned} \tag{240}$$

Thus, (218) and (239) coincide. ■

B.3 Dynamic System

$$\frac{\dot{C}}{C} = r - \rho. \tag{241}$$

$$\dot{A} = rA + wL + R\bar{Z} - pH - C, \tag{242}$$

$$\dot{H} = \tilde{B}^H \left(K^X\right)^{\beta\gamma} \left(L^X\right)^{\beta(1-\gamma)} \bar{Z}^{1-\beta} - \delta^H H, \tag{243}$$

$$\frac{\dot{q}^H}{q^H} + \frac{p}{q^H} = r + \delta^H. \tag{244}$$

$$C = \frac{pH}{\theta}, \tag{245}$$

$$A = K + q^H H, \tag{246}$$

$$K^X + K^Y = K, \tag{247}$$

$$L^X + L^Y = L, \tag{248}$$

$$K^X = \frac{\gamma}{1-\gamma} \frac{wL^X}{r + \delta^K}, \tag{249}$$

$$K = \frac{w}{r + \delta^K} \left(\frac{\gamma}{1-\gamma} L^X + \frac{\alpha}{1-\alpha} \left(L - L^X \right) \right), \tag{250}$$

$$w = (1-\alpha) B^Y \left(\frac{K - K^X}{L - L^X} \right)^{\alpha}, \tag{251}$$

$$q^H = \frac{(1-\alpha) B^Y \left(\frac{K-K^X}{L-L^X} \right)^{\alpha}}{(1-\gamma)\beta \tilde{B}^H \bar{Z}^{1-\beta}} \frac{\left(L^X\right)^{1-\beta(1-\gamma)}}{\left(K^X\right)^{\beta\gamma}}, \tag{252}$$

$$R^Z = (1-\beta) q^H \tilde{B}^H \left(\frac{\left(K^X\right)^{\gamma} \left(L^X\right)^{1-\gamma}}{\bar{Z}} \right)^{\beta}. \tag{253}$$

Recall that $\tilde{B}^H = B^H (B^X)^{\beta}$. Initial state variables are $K_0 = given$ and $H_0 = given$ and $A_0 = K_0 + q_0^H H_0$ (notice that A_0 is a jump variable). These are 13 equations and 13 endogenous variables: A, C, L^X, L^Y, K, K^X, K^Y, H, r, w, R^Z, p, q^H.