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PARETO SUPERIORITY OF UNEGALITARIAN EQUILIBRIA IN STIGLITZ' MODEL OF WEALTH DISTRIBUTION WITH CONVEX SAVING FUNCTION[1]

BY FRANÇOIS BOURGUIGNON

This paper extends the conclusions obtained by Stiglitz and others about the asymptotic wealth distribution in the neo-classical growth model when the saving function is convex. It is shown not only that locally stable two-class unegalitarian equilibria may exist along with the egalitarian equilibrium, but also that they necessarily are Pareto superior to it. More generally, the paper also analyzes the class of Pareto optimal unegalitarian equilibria.

1. INTRODUCTION

FEW ATTEMPTS HAVE BEEN MADE in the literature to get at a dynamic theory of income and wealth distribution integrating microeconomic models of accumulation and macroeconomic theories of factors' remuneration. Stiglitz' 1969 well known model [7] is one of them. Individuals accumulate wealth according to some conventional saving function. On the aggregate, this accumulation changes the remuneration rate of wealth relative to that of labor and this modifies the individual rates of wealth accumulation. In the neo-classical framework, it is shown that, with linear or concave saving functions (constant or decreasing marginal propensity to save), the distribution of income and wealth tends asymptotically toward equality.[2] With convex saving functions, Stiglitz indicates that the distribution of income and wealth might tend toward a "two-class" equilibrium but his analysis is not very detailed. The case of convex saving functions, however, is both relevant and important. Cross-sectional studies seem to indicate that the marginal propensity to save increases with income and/or wealth and that empirical fact is behind the commonly held view that income equality might conflict with growth and aggregate welfare (see [**2**, **4**, **5**], for instance).

A more detailed analysis of the convex saving function in Stiglitz' model has recently been proposed by Schlicht [**6**] who has shown that, under that condition, locally stable unegalitarian stationary distributions or 'equilibria', might exist along with the egalitarian one. Schlicht, however, did not consider the welfare implications of the coexistence of egalitarian and unegalitarian stable equilibria. Interestingly enough, it is proven in the present note that, when they exist, locally stable unegalitarian stationary distributions are *Pareto superior* to the egalitarian one. In other words, inequality in the neo-classical framework permits not only the achievement of higher aggregate income and consumption per capita as could have been expected, but also higher income and consumption for all individuals. It follows that, when savings is left to the private initiative and when

[1] This note is a revision of part of a paper presented at Pr. Malinvaud's CNRS Seminar. I thank the participants in that seminar and two anonymous referees for useful comments.

[2] Except, as noted by Stiglitz himself, when saving is zero below some minimum income level (see below).

unegalitarian asymptotic distributions exist, the optimal asymptotic distribution of income and wealth corresponding to any social welfare function of the utilitarian type is necessarily unegalitarian even though all individuals in the population are assumed to be identical. It must be stressed, however, that this result applies only to equilibria where all individuals have a positive wealth, which might not be the general case, especially in developing countries.

2. STIGLITZ' MODEL

Stiglitz' model may be summarized as follows. All individuals earn the same labor income w and receive a return on their wealth, c, at a rate of r. Their total current income is then

$$(1) \qquad y = w + rc.$$

The population is broken down into m groups i of individuals with the same wealth, c_i. Those groups grow at the same demographic rate, n. Therefore, they have a constant weight a_i in the population. They also are assumed to be closed to each other with respect to wealth transmission. Wealth is passed on from a generation to the next within a group and inheritances are assumed to be equally divided among heirs. The whole stock of capital is privately owned and the remuneration of labor and capital follow the neo-classical mechanism. We have then

$$(2) \qquad k = \sum_{i=1}^{m} a_i c_i, \qquad w = f(k) - kf'(k), \qquad r = f'(k),$$

where k is the stock of capital per capita and $f(k)$ is the conventional neo-classical aggregate production function.

All individuals have the same monotonically increasing saving function $S(y)$.[3] We will assume that $S(\ )$ is convex and that the marginal propensity to save tends toward one when income becomes infinitely large:

$$(3) \qquad 0 < S'(y) < 1, \qquad S''(y) > 0 \quad \text{and} \quad S'(y) \to 1 \quad \text{when} \quad y \to \infty.$$

Regrouping (1) and (2), the dynamic behavior of the economy and of the wealth distribution is fully described by the following differential system:

$$(4) \qquad \begin{cases} \dot{c}_i = S[f(k) + (c_i - k)f'(k)] - nc_i & (i = 1, 2, \ldots m), \\ k = \sum_i a_i c_i, \end{cases}$$

where $\dot{c}_i$ stands for dc_i/dt.

[3] S may also be assumed to depend on wealth (c) and the rate of interest (r) without modifications of the conclusions which follow. Schlicht [6] also considers the case of "relative income" saving functions. For the sake of simplicity, we restrict the present note to the case $S(y)$.

3. EGALITARIAN AND UNEGALITARIAN ASYMPTOTIC STATES

If the initial wealth is equally distributed, the growth path is egalitarian and converges toward the egalitarian equilibrium (EE) defined by:

$$c_i = k = k^0 \quad \text{with} \quad S\big[f(k^0)\big] = nk^0. \tag{5}$$

We shall assume that there is an unique locally stable egalitarian solution to (5).

ASSUMPTION 1: *The equation* $S[f(k)] = nk$ *has a unique root* k^0 *such that* $S' \cdot f'(k^0) < n$.

Letting I be the "stability interval" of k^0, we shall also assume that the "golden rule" capital-labor ratio k^*, $f'(k^*) = n$, belongs to I.

Indeed it seems reasonable to restrict the analysis of unegalitarian equilibria (UE) to initial situations (I) where the wealth distribution can actually converge toward equality. The assumption about k^*, on the other hand, will permit a sensible comparison of the economic efficiency of UE and EE.

Let us consider now growth paths starting from initial unegalitarian distributions. Possible asymptotic states or equilibria for such paths are given by the system:

$$\begin{cases} E(k,c_i) = S\big[f(k) + (c_i - k)f'(k)\big] - nc_i = 0 & (i = 1,2\ldots m), \\ k = \sum_i a_i c_i, \end{cases} \tag{6}$$

and by the usual local stability conditions. $S(\ )$ being convex, the equation $E(k,c) = 0$ can have only two roots in c so that there can be at most two classes of individuals at an UE. Under these conditions, we may as well restrict the analysis to the case where there are only two groups of individuals right at the beginning ($m = 2$).[4]

To study the existence of UE solutions to (6), consider the locus (E) in the space (k,c) defined by $E(k,c) = 0$. That equation also states:

$$f(k) + (c - k)f'(k) = T(nc) \tag{7}$$

where $T(\ )$ is the inverse functions of $S(\ )$ and is such that

$$T' \geqq 1, \qquad T'' < 0, \qquad T'(x) \to 1 \quad \text{when} \quad x \to \infty.$$

Along (E), we have

$$\frac{dc}{dk}\big[nT'(nc) - f'(k)\big] = (c - k)f''(k). \tag{8}$$

[4] If $m = 3$, it can easily be proven by looking at the Jacobian matrix of (6) that the rich class in a locally stable UE contains only one of the initial groups i.

From (6) or (7), it can be seen that (E) passes through (k^0, k^0) and, by Assumption 1 and (8) that the sign of dc/dk changes at that point, becoming positive when k gets larger than k^0. Consider now the curve (C) defined by

$$nT'(nc) = f'(k).$$

It is increasing and asymptotic to the golden rule k^*. It follows that the locus (E) defined by (7) has the shape depicted in Figure 1. One of its branches is asymptotic to k^*, it crosses vertically the curve (C) above the line $c = k$ and it crosses horizontally that line at k^0.

From Figure 1, it is clear that an UE will exist if it is possible to find (k, c') and (k, c'') on (E) such that $c'a_1 + a_2c'' = k$ (or $a_1c'' + a_2c' = k$). A first necessary condition for the existence of an UE is then: $k^0 < k^*$. Next, consider a value of

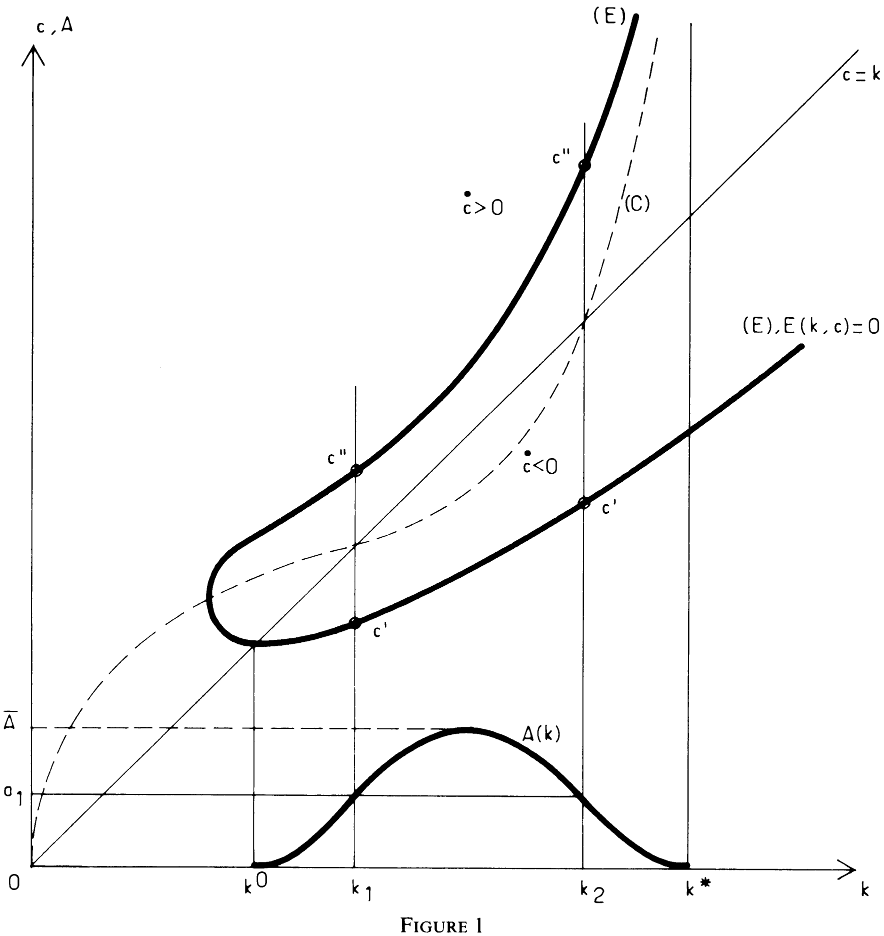

FIGURE 1

$k \in)k^0, k^*($ and the associated values $c'(k), c''(k)$ on (E). Let $A(k)$ be defined by

$$A(k)c'(k) + \left[1 - A(k)\right]c''(k) = k.$$

On Figure 1, we see that this function takes the value zero at both ends of $)k^0, k^*($ and, being continuous, it has a maximum, $\bar{A}$, on that interval. For an UE to exist at some $k \in)k^0, k^*($ we must then have

$$A(k) = a_1 \quad \text{or} \quad A(k) = a_2.$$

This is possible only if $\inf(a_1, a_2) \leqq \bar{A}$. So, one can see that *a necessary and sufficient condition for the existence of an unegalitarian equilibrium is*:

(a) $\quad k^0 < k^*, \quad \inf \ (a_1, a_2) \leqq \bar{A}.$

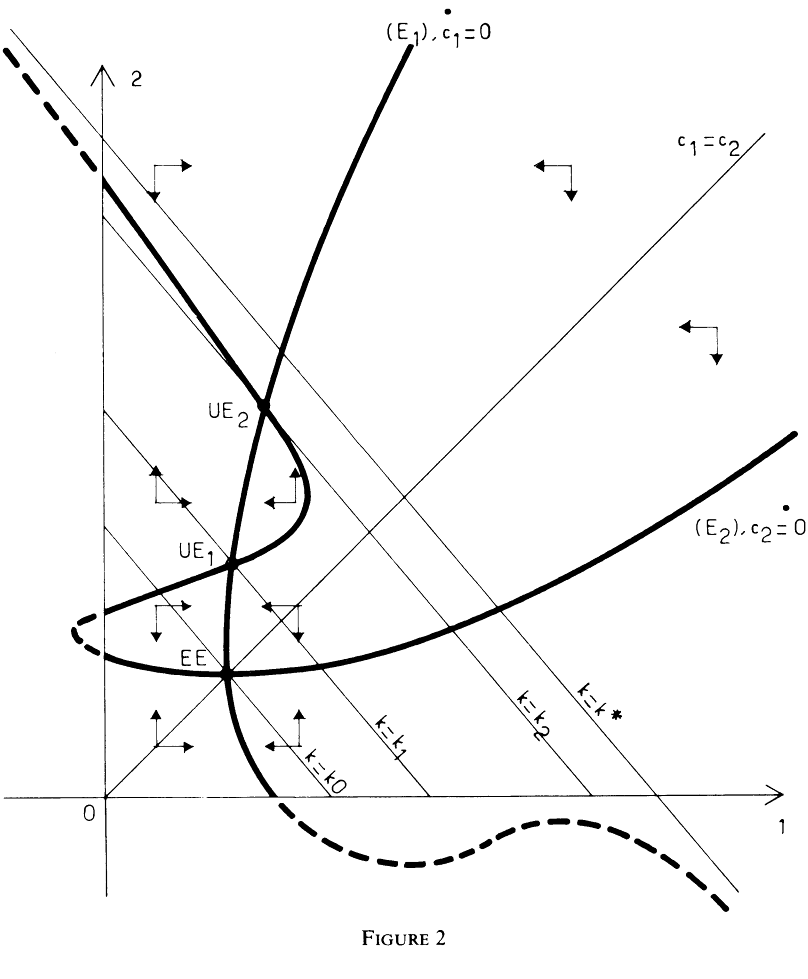

FIGURE 2

We can now get a little further and look at the local stability properties of the UE's. First, we can notice on Figure 1 that UE's necessarily go by pairs since there will always be an even number of solutions to the equation $A(k) = a_1$ (or a_2). As it is generally true that stable steady states alternate with unstable ones, k_1 on Figure 1 will likely correspond to an unstable UE whereas k_2 will correspond to a stable UE. To check that point, it is sufficient to translate Figure 1 into the space (c_1, c_2). In that space, there will be two curves (E) corresponding respectively to $c_1 = 0$ and $c_2 = 0$. Considering the positive orthant only and noticing that (4) implies $\dot{c}_i < 0$ between the two branches of the corresponding (E) locus, we get the trajectory directions shown on Figure 2. As expected, the unstable steady-state at k_1 lies between the stable EE at k^0 and the stable UE at k_2. So, the conditions (9) are in fact necessary and sufficient for the existence of a *locally stable* UE.

4. PARETO SUPERIORITY OF UNEGALITARIAN EQUILIBRIA

The Pareto superiority of the UE's with respect to the EE at k^0 can be simply derived from Figure 1 and the concavity of $f(\ )$. The latter property implies:

$$(10) \qquad f(k) + (c - k)f'(k) > f(c) \qquad \text{for any} \quad (k, c).$$

The left-hand term of that equation is simply the income of individuals with wealth c when the average wealth in the population is k. On the other hand, it can be seen on Figure 1 that any UE is such that

$$(11) \qquad k^0 < c'(k) < c''(k).$$

Together with (10), this proves the Pareto superiority of unegalitarian equilibria over the egalitarian equilibrium.[5]

Comparing now unegalitarian equilibria among themselves, the same reasoning shows that the stable UE at k_2 on Figure 1 is Pareto superior to the unstable UE at k_1. It must be noticed, however, that a stable UE between k_2 and k^* would not be Pareto superior to the UE at k_2 although c' and c'' would be larger. This is simply because that UE would correspond to a different distribution (a_1, a_2) of the population among the rich and the poor class. So, going from k_2 to some larger value of k would involve some individuals falling from the rich class to the poor class and consequently, being worse off. Although inequality permits getting closer to the golden rule and increasing any utilitarian social welfare function, it is not true that "the closer to the golden rule, the better." We can only say that, in presence of a convex saving function and under conditions (9), Pareto optimal equilibria necessarily are stable UE at the right end of the interval (k^0, k^*).[6]

[5] This argument shows that the income of both the rich and the poor class are larger at an UE than at the EE. Assuming that the marginal propensity to save is less than one (3), this is also true for consumption.

[6] This is not true of *all* stable UE since the poor class at an UE might be richer than the rich class at another UE with a smaller k.

The preceding conclusions about the Pareto optimality of the UE refer to the case where all individuals have a strictly positive wealth. If the analysis is not restricted to the stability interval, I, of the EE, one may follow Stiglitz in considering unegalitarian equilibria where a majority of individuals have a zero wealth because their labor income hardly covers their basic needs. It can be proven that those UE correspond to a capital-labor ratio outside of I and that they are definitely Pareto inferior to the EE. In a dynamic framework they are not necessarily sub-optimal, however, since inequality might be the only way to prevent the decay of the economy outside of I.

It might be argued, finally, that Stiglitz' framework with static saving functions is not the most appropriate for normative conclusions and that the inter-temporal nature of the saving process should be taken into account as in Atkinson [**1**] or Hamada [**3**]. As long as we limit ourselves to stationary distributions, however, it is most probable that the preceding results will generalize to inter-temporal saving functions.

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