The Pasinetti Paradox in Neoclassical and More General Models [1]

I. INTRODUCTION

In 1962 Dr Pasinetti [2] enunciated a remarkable result.

Pasinetti's Theorem:

Consider a system in which the labour force grows at some exponential rate n', technological progress is Harrod-neutral and occurs at a constant rate n'', and therefore the " effective " labour supply, measured in " efficiency " units, rises at the rate $n = n'+n''$.[3] Suppose further that one can meaningfully identify a class of income receivers—the " capitalists "—whose *sole* source of income is earnings from capital, and suppose that this group has a propensity to save (average and marginal) equal to s_c. Call the remainder of the community " workers " and assume they save the fraction s_w of their wage or interest income.

Then, if the system is capable of generating a " golden-age " growth path along which income, consumption and capital all grow exponentially at the " natural rate " n, this equilibrium growth path has the following remarkable properties:

1. The steady-state rate of return to capital or rate of interest, r^*, depends only on the rate of growth n and on s_c according to the simple formula $r^* = n/s_c$; it is therefore *completely independent* of the workers' saving propensity s_w or of the form of the production function.

2. The steady-state capital output ratio $(K/Y)^*$, the capital labour ratio $(K/L)^*$ and the share of income going to capital $\alpha_k^* = (rK/Y)^*$ are also independent of s_w; again, they depend on n/s_c, but also on the *form* of the production function.

From this surprising theorem come other remarkable corollaries such as: if s_c is unity, a situation that Pasinetti associates (pp. 277-278) in particular with a socialist state —although the association is questionable, as indicated below—then it follows from (1) that r will tend to the natural rate of growth n, and saving will approach the Swan-Phelps [4] golden rule of accumulation $S = I = P$, where $P = rK$ is total income from capital.

A result of this generality puts us all in debt to Dr Pasinetti, and is worthy of further study and elucidation, which is what we offer in this paper.

[1] We should like to acknowledge helful research assistance from Felicity Skidmore, and financial aid from the Ford Foundation and the Carnegie Foundation. Helpful comments were received from the Harvard-M.I.T. mathematical economics colloquium. In particular we are indebted to R. Solow and G. LaMalfa for helpful criticism.

[2] Luigi L. Pasinetti, " Rate of Profit and Income Distribution in Relation to the Rate of Economic Growth ", *Review of Economic Studies*, **29** (1962), 267-279, a seminal paper.

[3] To sidestep the complications that come in when K/L goes asymptotically to zero or infinity (when capital is incapable of being widened fast enough or slow enough), we first posit that positive labour is needed to produce positive output, and that n is positive but not too large. We remove this simplifying restriction in the Appendix which gives an exhaustive analysis of possible pathological divergences.

[4] T. W. Swan, " Of Golden Ages and Production Functions ", presented at the Round Table on Economic Development in East Asia (International Economic Association), Gamagori, Japan, April 1960 (revised 1962), mimeo., 18 pp.; E. S. Phelps, " The Golden Rule of Accumulation ", *American Economic Review*, **51** (1961) 628-643; J. Robinson, " A Neoclassical Theorem ", *Review of Economic Studies*, **29** (1962), 219-226, with Comments by R. M. Solow, P. A. Samuelson, J. E. Meade *et al.*; C. C. von Weizsäcker, *Wachstum, Zins und Optimale Investitionsquote* (Kyklos, Verlag, Basel, 1962), 96 pp.

(*a*) First, we shall show the limited range of the workers and capitalists saving coefficients within which the above formulation of the Pasinetti Theorem is valid. Outside that range we formulate a theorem that is dual to it and of the same generality. It too involves a paradox, namely that the average product of capital—the reciprocal of the capital output ratio—to which the system settles is, this time, equal to n/s_w and completely independent of the s_c propensity to save out of profit or of the form of the production function. All the other golden-age variables of the system depend only upon n/s_w and on the form of the production functions. The complete duality of all this with the Pasinetti theorem is notable and we state the general case that covers all contingencies.

(*b*) A second, though relatively minor purpose, is to help dispel the notion, which seems to have been entertained by some readers, that Pasinetti's analysis has some peculiar relevance to a Kaldorian *alternative* theory of distribution, of the type presented by Kaldor in 1955, 1957, 1961, and 1962, or to some version of a " Cambridge " theory of distribution.[1] The following lead sentence of Pasinetti (and indeed his whole lead paragraph) might predispose the reader in the street to this view.

" One of the most exciting results of the macro-economic theories which have recently been elaborated in Cambridge is a very simple relation connecting the rate of profit and the distribution of income to the rate of economic growth, through the interaction of the different propensities to save " (p. 267).

Other passages may give the impression that the major accomplishment of his analysis is to remove " a logical slip " (p. 270) in the Kaldorian formulation which allowed for some saving by workers but did not permit these savings to accumulate and produce income.[2]

Actually, as Dr Pasinetti makes clear at many places (and in particular in the long footnote on p. 276) his analysis is one of the greatest generality. His theorem applies in fact to any system capable of a golden-age growth path. But, precisely because of this great generality, his analysis can in no way help us to discriminate between alternative theories of income distribution. In order to make this point perfectly clear, and for its own sake, our own analysis shall deal primarily with a neoclassical production function capable of smooth factor substitution and with the case of perfectly competitive markets. Later we shall provide some indications of how the analysis might be extended if either of these conditions fails.

(*c*) Finally, we shall investigate, and prove, the stability of the Pasinetti golden-age in the case where it is valid. That is, we shall prove that the system will asymptotically approach the steady-state from arbitrary initial conditions, at least in a local neighbourhood of that unique state. And where our anti-Pasinetti golden age holds, in which the workers' saving propensity is all important, we demonstrate its global asymptotic stability.

Our asymptotic stability analysis, which can be extended in considerable measure to certain cases of fixed-proportions and distribution theories different from that of

[1] N. Kaldor, " Alternative Theories of Distribution ", *Review of Economic Studies*, **23** (1955), 83-100 (reprinted in *Essays on Value and Distribution*, pp. 228-236); " A Model of Economic Growth ", *Economic Journal*, **67** (1957), 591-624 (reprinted in *Essays in Economic Stability and Growth*, pp. 256-300); F. Lutz and D. C. Hague (eds.), *The Theory of Capital* (1961), pp. 177-220, from the 1958 Corfu I.E.A. Conference; with J. A. Mirrlees, " A New Model of Economic Growth," *Review of Economic Studies*, **29** (1962), 174-192. Cambridge writings related to Kaldorism, but not necessarily identical with it, are J. Robinson, *Accumulation of Capital* (1956) and *Collected Economic Papers*, **2** (1960), particularly pp. 145-158, " The Theory of Distribution "; *Exercises in Economic Analysis* (Macmillan, London, 1960), *Essays in the Theory of Economic Growth* (Macmillan, London, 1962); D. G. Champernowne, " Capital Accumulation and the Maintenance of Full Employment ", *Economic Journal*, **67** (1958), 211-244; R. F. Kahn, " Exercises in the Analysis of Growth ", *Oxford Economic Papers*, New Series, **10** (1958), 143-156.

[2] Actually, it might be argued that there need not be a " logical slip " in the Kaldorian model, if it will merely assume that the propensity to save out of income from capital is s_c whether that income is received by capitalists or by workers. This hypothesis, which may or may not be empirically sound, is certainly not *logically* self-contradictory—as our colleague Professor Solow has pointed out to us. Actually, such a model has been extensively studied by many writers as can be seen by looking up the references Uzawa (1961), Solow (1961), Inada (1963), Drandakis (1963) and still others in the valuable bibliography to F. H. Hahn and R. C. O. Matthews, " The Theory of Economic Growth: A Survey ", *Economic Journal*, **74** (1964), 779-902.

marginal productivity, must not be confused with the problem of stability of the instantaneous differential equations of capital formation and growth at continuous full employment (or at any other posited level of employment). Dr Pasinetti's stability analysis seems to be related to the instantaneous rather than asymptotic state. In the context of fixed-coefficient models, such stability analysis becomes very intricate indeed. We shall not here attempt to clarify the real and formidable problems posed by such a model for theories of distribution (and/or full employment) of macroeconomic type.

Our general analysis is shown to apply even though capitalists and workers may be divided into any number of subcategories each with a different propensity to save. On this growth path the rate of interest, the capital-output and the capital-labour ratio always depend at most on but one of the various capitalists' propensities to save (the maximum one), and are completely independent of all the others.

In view of the many nice properties sketched out above which hold for an economy satisfying the saving assumptions of the present model, it is with some regret that we must confess to most serious qualms over the empirical relevance of these assumptions—notably that relating to the existence of identifiable classes of capitalists and workers with " permanent membership "—even as rough first approximation. These qualms and the grounds on which they rest are set forth in the concluding section.

II. THE NEOCLASSICAL FORMULATION OF THE MODEL

Let total real output be produced by labour L and by total physical homogeneous capital K.[1] Total K is split into two parts, K_c the capital of the " capitalist " class, and K_w, the capital of the class which receives at least part of its income from labour.

For simplicity, the production of consumption output C and of net capital formation $\dot{K} = dK/dt$ is assumed to involve the same capital-labour factor intensities; this conventional Ramsey-Solow simplification means that total real net output Y can be split up into real consumption C plus real net capital formation $\dot{K}$, and for proper choice of units can be written as $Y = C + \dot{K}$. All this is summarized in the relation

$$Y = C+\dot{K} = F(K_c+K_w, L), \text{ a constant-returns-to-scale function}^{2}$$

$$K = K_c+K_w, \dot{K} = \dot{K}_c+\dot{K}_w. \qquad \ldots(1)$$

For analytical purposes we find it frequently convenient to deal with variables expressed per head of the " effective " labour force L; variables so expressed will be denoted with the same letter as the corresponding aggregates but in lower case. Thus $y = Y/L$, $k = K/L$, $k_c = K_c/L$, etc. With this notation, equation (1) in view of its first-degree homogeneity property, can be rewritten as

$$y = F(k, 1) = f(k), \quad \frac{\partial F(k, 1)}{\partial k} = f'(k) > 0 \qquad \ldots(1')$$

with $\dfrac{d^2f(k)}{dk^2} = f''(k) < 0$ in consequence of " diminishing returns ". We shall posit neoclassical smoothness and substitutability and perfect markets, under which conditions competition will *enforce* at all times equality of factor prices to factor marginal productivities,[3] namely,

1 Heterogeneity of capital goods can, under certain special assumptions like those underlying surrogate capital models, be introduced without necessarily vitiating the results.

2 The assumption of constant returns to scale is essential both to Pasinetti's and our own analysis, for otherwise the concept of a golden age steady state becomes self-contradictory. If depreciation is mK, then $F(K, L)+mK$ is the function for *gross* national product.

3 Of course w is the wage rate per efficiency unit if Harrod-neutral technical change is going on. As we shall see in Section 10 below, for most of our results it is not necessary that r and w be equal to the marginal product of capital and labour.

$$\text{interest or profit rate} = r = \frac{\text{Total Profits}}{\text{Total Capital}} = \frac{P}{K} = \frac{P_c+P_w}{K}$$

$$= \text{marginal product of capital} = \frac{\partial F(K, L)}{\partial K} = f'(k)$$

$$\text{real wage rate} = w = \frac{\text{Total Wages}}{\text{Labour}} = \frac{W}{L} = \frac{Y-rK}{L}$$

$$= f(k)-kf'(k). \qquad \ldots(2)$$

We shall also denote the average product of capital $Y/K = f(k)/k$ by $A(k)$, which is the reciprocal of the capital-output ratio K/Y, and note for later reference that, with a well-behaved production function, i.e. with concave $f(k)$ having the properties $f'(k)>0$, $f''(k)<0$, k is a monotonic decreasing function of r, while the average product $A(k)$ is a decreasing function of k, i.e. $A'(k)<0$. Finally we shall use the symbol $\alpha(k)$ to denote the share of income accruing to capital, for given k, i.e. $\alpha(k) = rk/f(k)$, which reduces, when marginal productivity relations are valid, to the ratio of marginal to average product $f'(k)/A(k)$.

Now, the basic savings-investment equations for the two classes—which hold for all time periods, short or long—can be written down:

$$\dot{K}_c = s_c P_c = s_c(rK_c) = s_c K_c \frac{\partial F(K_c+K_w, L)}{\partial K}$$

$$\dot{K}_w = s_w(W+P_w) = s_w(W+rK_w) = s_w(Y-rK+rK_w) = s_w(Y-rK_c)$$

$$= s_w\left[F(K_c+K_w, L)-K_c\frac{\partial F(K_c+K_w, L)}{\partial K}\right]. \qquad \ldots(3)$$

The first equation says that the total saving of the capitalist class, and hence the rate of growth of their capital K_c, equals s_c times their total profits. The second equation says that workers savings, and hence the rate of growth of their capital K_w, is a fraction of s_w of their total income, consisting of wages W and income from their capital, rK_w, or equivalently of total income less capitalists income.

Equations (1), (2) and (3), or the equations (3) alone in their final form, give us a determinate growth system in (K_c, K_w) once we are given the time profile of labour employment $L(t)$. Now we posit the usual exponential growth of the labour force, which we equate with labour input $L(t)$, implying $L(t) = L_0e^{nt}$ (with the understanding that the natural rate of growth n could include Harrod-neutral technical change, in which case L must be given an efficiency-unit interpretation). Hence

$$L = L(t) = L_0e^{nt}, \quad \frac{\dot{L}}{L} = n$$

$$\frac{\dot{k}}{k} = \frac{(\dot{K/L})}{K/L} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - n \qquad \ldots(4)$$

$$\frac{\dot{k}_c}{k_c} = \frac{\dot{K}_c}{K_c} - n, \quad \frac{\dot{k}_w}{k_w} = \frac{\dot{K}_w}{K_w} - n.$$

Using the equations (3) to substitute for $\dot{K}_c/K_c$ and $\dot{K}_w/K_w$ in the right-hand side of the last two equations and writing k as short for k_c+k_w, we obtain

$$\frac{\dot{k}_c}{k_c} = s_c r - n = s_c f'(k) - n$$

$$\frac{\dot{k}_w}{k_w} = s_w \frac{Y - rK + rK_w}{K_w} - n = s_w \frac{f(k) - rk}{k_w} + (s_w r - n)$$

$$= s_w \frac{A(k) - f'(k)}{k_w} k + [s_w f'(k) - n] = s_w \frac{f(k) - k_c f'(k)}{k_w} - n, \qquad \ldots(5)$$

a system of two simultaneous differential equations in the intensive variables k_c and k_w, and not explicitly containing any dependence on time.

For convenient future reference, we can combine the equations of (5) to show, after various substitutions, that

$$\frac{\dot{k}_c}{k_c} - \frac{\dot{k}_w}{k_w} = \frac{(\dot{k_c/k_w})}{(k_c/k_w)} \gtreqless 0$$

depending upon whether $\qquad \ldots(5')$

$$\frac{k_c}{k_w} \lesseqgtr \frac{\alpha(k) s_c - s_w}{[1-\alpha(k)] s_w}.$$

This criterion is remarkable in that it does not contain n explicitly.

III. POSITIVE STEADY STATE SOLUTION

To find the steady state equilibrium values of k, k_w and k_c, say (k^*, k_w^*, k_c^*), we set $\dot{k}_c$ and $\dot{k}_w$ equal to zero in (5) and solve the static equations

$$s_c f'(k) - n = 0$$

$$s_w[f(k) - rk] + (s_w r - n) k_w = 0,$$

or

$$f'(k^*) = r^* = \frac{n}{s_c}$$

$$k_w^* = s_w \frac{A(k^*) - f'(k^*)}{n - s_w f'(k^*)} k^* = \frac{\text{av. product of capital} - \text{marg. product of capital}}{\frac{n}{s_w} - \frac{n}{s_c}} k^*$$

$$k_c^* = k^* - k_w^* = \frac{n - s_w A(k^*)}{n - s_w f'(k^*)} k^* = \frac{\frac{n}{s_w} - A(k^*)}{\frac{n}{s_w} - \frac{n}{s_c}} k^* \qquad \ldots(6)$$

$$\frac{k_c^*}{k_w^*} = \frac{\alpha(k^*) s_c - s_w}{[1-\alpha(k^*)] s_w}.$$

The first of these equations (6) will be recognized as Pasinetti's Theorem: on the equilibrium growth path (where $\dot{K}/K = \dot{K}_c/K_c = \dot{K}_w/K_w = n$), the rate of interest is determined by n and s_c only. To this value of r^* there corresponds in turn the unique capital-labour ratio k^*, the unique average product of capital $A(k^*)$, and its unique reciprocal, the capital output ratio $k^*/f(k^*)$. Hence these ratios are also independent of s_w; but still they do depend on the form of the production function F or f. Note also that all solutions of (6) depend on (n, s_w, s_c) only in the ratio form $(s_c/n, s_w/n)$.

However for the solution (6) to be economically meaningful it must satisfy the non-negativity condition $k_w \geqq 0$, $k_c \geqq 0$. The implication of the first inequality can be inferred from the second of the equations (6). Since the numerator of the expression on the right-hand side is the difference between the average and the marginal product of capital, which is necessarily non-negative, the denominator of the fraction must be positive if k_w is to be non-negative and finite or,

$$s_w < s_c. \qquad \ldots(7)$$

i.e. the workers' saving propensity must be smaller than that of the capitalists if Pasinetti's theorem is to hold. This same inequality is implicit in Pasinetti's inequality conditions (6) and (7). But we must hasten to add that, though (7) is necessary for Pasinetti's theorem (as we have characterized it) to apply, and though (7) is necessary for many versions of the Kaldorian theory of income distribution to yield economically meaningful results, (7) has in general nothing to do with the existence and stability of a steady-state full employment equilibrium, as we shall presently show explicitly.

As for the second non-negativity condition $k_c \geqq 0$, we can see from (6)'s last equation, derivable from (6)'s other equation or from the criterion at the end of the last section, that

$$s_w \leqq \alpha(k^*)s_c = n\frac{k^*}{f(k^*)}. \qquad \ldots(8)$$

This inequality is seen to be more stringent than (7) since the capital share $\alpha(k)$ is generally less than one, and empirically very much less than one. Thus if $\alpha(k^*) = \frac{1}{4}$, and $s_c = \frac{1}{5}$, Pasinetti's theorem could not hold for s_w any higher than a modest 0·05.[1]

Inequality (8) has some correspondence to Pasinetti's (6), which says $s_w < \frac{I}{Y}$, since in steady-state equilibrium when $I/K = \dot{K}/K = \dot{L}/L = n$, we have $I/Y = nK/Y = n\frac{k}{f(k)}$. However, Pasinetti's simple inequality[2] is ill-defined, since outside of the steady-state equilibrium, I/Y could take any value whatever; and even on the equilibrium growth

[1] These numbers are econometrically reasonable for a mixed economy like the U.S., U.K., or Western Europe. Rather different-appearing numbers would seem to come from an argument like the following. " Suppose corporations pay out in dividends only $\frac{2}{3}$ of their earnings (which constitute most of the earnings of capital) and individuals all save about one-twelfth of their disposable incomes. (Such numbers are econometrically not too unrealistic.) Then identify $s_c = 1-\frac{2}{3} = \frac{1}{3}$, $s_w = \frac{1}{12}$, and find that we can stay in the Pasinetti regime if α_K is about one-fourth." Actually, of course, the above behaviour equations would lead to the old Kaldor model whose logical consistency is criticized by Pasinetti and defended, as a possibility in our footnote 6. So, sticking to the notion that only one saving propensity applies to the worker class regardless of type of income, we would have to reinterpret the above data as follows. If people know that along with each \$2 of dividends they receive, there is saved for them \$1 in ploughed-back corporate earnings and that this can with reasonable confidence be deemed to yield them equivalent (and lightly taxed!) capital gains, they will (and there is some econometric evidence that they actually do) include their share of imputed corporate income in their true income and will adjust accordingly (although perhaps not on a 100 per cent basis) their saving out of so-called disposable personal income. Hence, a pure capitalist may prudently spend $\frac{6}{5}$ of his dividends and still end up saving $\frac{1}{5}$ of his true imputed income. If workers are stubbornly to end up with the same s_w for their share of capital income—a somewhat implausible hypothesis—they must spend $\frac{57}{40}$ of their dividends to keep $s_w = \frac{1}{20}$. Who will buy the stock that some people are liquidating? Anyone who fully understands the meaning of equation (3) will know the answer: the saving out of wages will be just enough in the model to match the overspending out of dividends. Repeatedly we give our reservations about the realism of the strict Pasinetti assumptions. (Warning: if one tries to oversimplify reality by forcing it into the mould of simple propensities to save of people and corporations, one should realize that a corporate propensity to save of $\frac{1}{3}$ reflects in real life Kuh-Meyer effects in which—to oversimplify reality again—*all* of corporate investment is not autonomous with corporate saving independent of it. E.g., if a corporation has a marginal-propensity-to-invest its " cash " earnings of $\frac{1}{6}$ and a marginal-propensity-to-save of $\frac{1}{3}$, then in all Keynesian multiplier formulas for effective demand, the relevant marginal leakage coefficient is not $\frac{1}{3}$ but rather $\frac{1}{3}-\frac{1}{6}$. One gets bad realistic prediction about comparative statics of mixed capitalism if one fails to take these interconnections into account—preferably in a less crude manner than described here.)

[2] After further discussion of the crucial limits on s_w/s_c, footnote 11 will return to the meaning of Pasinetti's (6).

path $f(k)/k$ is not a given of the problem but a characteristic of the solution, if any, except possibly in the very special case of fixed production coefficients, where K/Y might be identified with the technologically determined (minimum) capital coefficient. Our (8) has the merit of making explicit what must not be left ill-defined, namely that the inequality $s_w < \alpha(k^*)s_c$ must hold precisely at $k = k^*$, the k that corresponds to $r^* = n/s_c$. In what follows by r^*, k^* and other starred symbols we always mean the magnitude that corresponds to the root of Pasinetti's equation $f'(k) = n/s_c$, an equation that must be distinguished from the theorems that can sometimes be related to it.

IV. LIMITS ON THE PASINETTI THEOREM

To understand a theorem you must understand its limitations. The numerical range of the parameter s_w for which Pasinetti's theorem is applicable is severely limited, as indicated by (8). Let us see why, in terms of the comparative static properties of equations (6).

First consider s_c positive and s_w zero. Then freeze n and s_c, and consider the implication of higher values of the parameter s_w. For s_w zero, since both (7) and (8) are satisfied, we shall have $r^* = n/s_c$ and $k = k^*$ corresponding to r^*. Furthermore, $k_w^* = 0$ if $s_w = 0$, and hence $k_c^*/k^* = 1$.

Now let workers become thrifty. At first positive s_w will continue to satisfy (7) and (8) and therefore r^* and k^* will be unchanged. That is Pasinetti's remarkable theorem. Clearly k_w^* has become positive, showing that k_c^* must at first be forced down by rising s_w. Thrift on the part of workers, as long as s_w is sufficiently small, gives society no lasting appreciable *per capita* benefit; it merely causes capitalists of unchanged thriftiness to end up with less of the unchanged *per capita* wealth.

How does this come about dynamically? Start out in (r^*, k^*) equilibrium with $s_w = 0$. Now let s_w suddenly move to a permanent positive level, though still small enough to satisfy (8). The new flow of workers' saving will transiently increase k above k^*, decreasing r below r^*. Even though Y begins transiently to grow faster than n so that y rises, the lower interest rate means that the capitalists' K_c grows more slowly than before, which means more slowly than n. Hence, k_c initially drops *below* the old $k_c^* = k^*$. Once this is understood, it is easy to see that k_c will not continue *permanently* to fall but will instead approach the new critical level for k_c^* given by the next-to-the-last equation (6). It will have permanently declined to this level only when k_w^* has permanently risen to its new appointed level as given by the proper equation of (6). (Note: there could be damped oscillation around the new equilibrium, as will be shown later.)

Fig. 1 illustrates this by exhibiting the behaviour of k^∞ and k_c^∞—the values of k and k_c on the golden age path to which the system tends as $t \to \infty$—in function of s_w, and for a fixed value of n and of s_c. For concreteness, we have frozen s_c at $\frac{1}{5}$ and assumed $\alpha(k^*) = \frac{1}{4}$. The dashed locus, beginning as a straight line parallel to the abscissa is the graph of k^∞. Within the range of applicability of Pasinetti's theorem, i.e. $s_w < \alpha(k^*)s_c = 0{\cdot}05$, we know k^∞ is a constant k^*. The behaviour of k_c^∞ is shown by the declining solid curve starting at k^* on the abscissa. Within the range of validity of Pasinetti's theorem we know that $k_c^\infty = k_c^*$ and hence the equation of its locus is given by the second of equations (6), which can be conveniently rewritten as

$$k_c^* = \frac{1 - s_w \alpha(k^*)^{-1} s_c^{-1}}{1 - s_w s_c^{-1}} k^* = \frac{1 - 20 s_w}{1 - 5 s_w} k^*.$$

For $s_\omega = 0$, $k_c^* = k^*$. But as s_ω rises k_c^* becomes a smaller and smaller fraction (and k_w^* becomes a growing fraction) of the unchanging k^*, until for $s_w = 0{\cdot}05$ k^* reduces to zero and $k_w^* = k^*$. Here the applicability of Pasinetti's theorem and hence of equations (6) ends. The dotted continuation of the graph of the second equation (6) into the negative quadrant is of no economic significance. Instead, as we shall presently see, the valid extension of the k_c^∞ curve is the heavily-shaded horizontal axis itself. Similarly we will see that for $s_w > \alpha(k^*)s_c$, $k^\infty = k_w^\infty$ will rise above k^*, increasing monotonically with s_w, as shown by the rising portion of the dashed curve labelled k^{**}. It will be presently shown that the behaviour of k^∞, k_c^∞ and k_w^∞ in this range is covered by a theorem complementary to Pasinetti's theorem, its dual. The general theorem which covers both cases will demonstrate the remarkable duality results: for s_w on the left of the dividing line, $k^\infty = k^*$,

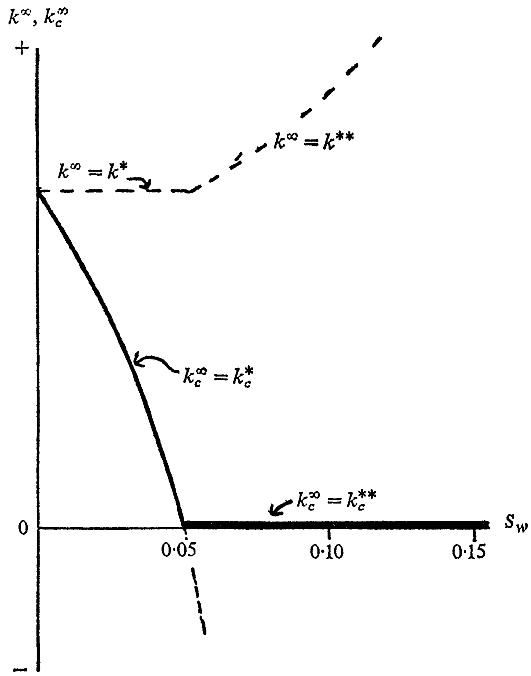

FIGURE 1

Behaviour of k^∞ and k_c^∞ for fixed n (assuming $\alpha(k^*) = \frac{1}{4}$; $s_c = \frac{1}{5}$).

the value determined by the condition that the *marginal product of capital must be equal to* n/s_c (and hence k^∞ is totally independent of s_w). But, on the other side of the dividing line $k^\infty = k^{**}$, the value determined by the condition that the *average product of capital must be equal to* n/s_w (and hence k^∞ is quite independent of s_c). The system in its wisdom will pick out of the dual regimes the one that gives it *most per capita* income and capital.

V. THE FUNDAMENTAL DUALITY THEOREM

To establish these results let us examine closely just what happens when condition (8) fails to hold, i.e. when

$$s_w > n\frac{k^*}{f(k^*)} = \frac{n}{A(k^*)} = \alpha(k^*)s_c. \qquad \ldots(9)$$

Dr Pasinetti has not explored this case although several passages in his paper suggest that the system would not tend to a steady state and in fact might even be incapable of maintaining full employment.[1] It should be immediately apparent that, as long as the production function is well behaved, failure of (8) or even of (7) to hold cannot interfere with full employment, which is insured by conditions (2). What happens instead is that, eventually the rate of growth of capitalists' assets will become and remain smaller than the rate of growth of workers' assets and also smaller than n. This in turn means that, asymptotically, k_c as well as K_c/K, the capitalists' share of total wealth, will approach zero while the workers' share, K_w/K will approach unity.

This conclusion may be seen in many ways, the most conclusive involving examination of the mathematical differential equations (5). The economic common sense is also very clear. For we have seen that as s_w rises from zero through small positive values, k_c^* is forced down to make room for growing k_w^*. What happens when k_c^* has been forced down to zero, which (6) shows takes place at the critical limit where (8)'s inequality begins to take over?

An increase in s_w beyond $\alpha(k^*)s_c$ must inevitably result in a decline of K_c/K and k_c toward zero. This is seen immediately in the extreme case where s_w exceeds s_c, since then the workers' savings out of profits *alone* will grow at a faster rate than can the total of all capitalists' savings. In the intermediate case where s_w is between s_c and $\alpha(k^*)s_c$, the same result can be demonstrated as follows. Begin in the Pasinetti k^* point with some positive k_c: then K_c is growing exactly as fast as n by definition of that point. However, it is now impossible that K_w and K should be growing as slowly as n at this point; the criterion (5′)'s numerator shows that, for any positive K_c/K_w level, (9) implies that K_w is growing faster than K_c, and hence both K_w and K are growing faster than n at the Pasinetti point. Hence, k grows beyond k^*, and r falls below r^*, now dictating a definite falling off of k_c toward zero. But despite the decline in k_c, total capital k must continue to grow past k^*. This conclusion can be established from the following equation for $\dot{k}$ which is implied by the two equations (5):

$$\dot{k} = \left(\frac{s_w}{n} - \frac{k}{f(k)}\right) nf(k) + rk_c(s_c - s_w). \qquad \ldots(5'')$$

In the first place this equation confirms that, when (9) holds, $\dot{k}$ is positive at k^* since

$$\frac{s_w}{n} > \frac{k^*}{f(k^*)} = \alpha(k^*)\frac{s_c}{n}.$$

It also shows that $\dot{k}$ must remain positive and hence k must continue to grow at least until it has become large enough for the first term to vanish, i.e. until $\frac{k}{f(k)} = \frac{s_w}{n} > \frac{k^*}{f(k^*)}$. But since for $k>k^*$ we must have $r<r^*$, k_c must continue to fall indefinitely, fading toward zero. The limiting value to which k tends, say k^{**}, can then be inferred from (5″) by setting $\dot{k}$ equal to zero and disregarding the last term. We thus find that k^{**} must satisfy the condition $\frac{k^{**}}{f(k^{**})} = \frac{s_w}{n}$. Or, as we prefer to express it in order to bring out the duality of our theorem to Pasinetti's, the system must approach the equilibrium $k \rightarrow k^\infty = k^{**}$

[1] ". . . if (6) [the equivalent of our (8)] were not satisfied the system would enter a situation of chronic Keynesian unemployment. Similarly if (7) were not satisfied the system would enter a situation of chronic inflation " (p. 269). Actually all that one can say is that I/Y must always be a weighted mean of the non-negative (s_c, s_w) coefficients, never lying outside their range because neither factor share can be negative. $I/Y = s_c \neq s_w$ would imply zero wage share, and $I/Y = s_w \neq s_c$ would imply zero profit share. But that is as far as mere arithmetic can take us. To infer inflation from $s_c < I/Y$ and unemployment from $s_w > I/Y$ requires behaviour hypotheses of a particular and special sort. Thus, in some of our neoclassical models characterized by inflationless full employment, golden ages emerge with $s_w = I/Y > s_c$.

where k^{**} is the root of the dual relation

$$A(k^{**}) = \frac{f(k^{**})}{k^{**}} = \frac{n}{s_w}. \qquad \ldots(10)$$

This result is readily understandable by noting that, with k_c tending to vanish, the limiting behaviour of our system reduces to the familiar Solow process with a single class of savers, namely the workers. It is well known that such a process deepens capital to an asymptotic limit given by the Harrod-Domar equation $\frac{K}{Y} = \frac{k}{f(k)} = \frac{s}{n}$, which agrees with our result above.

In summary, a system with $s_w > \alpha_k(r^*)s_c$ is bound to have the following asymptotic properties [1]

$$\lim_{t\to\infty} \frac{K_w}{K} = 1, \quad \lim_{t\to\infty} \frac{K_c}{K} = 0$$

$$\lim_{t\to\infty} \frac{\dot{K}_w}{K_w} = \lim_{t\to\infty} \frac{\dot{K}}{K} = n > \lim_{t\to\infty} \frac{\dot{K}_c}{K_c}$$

$$\lim_{t\to\infty} (k_c, k_w) = (0, k^{**})$$

$$\lim_{t\to\infty} k = k^{**}, \text{ the root of } \frac{f(k^{**})}{k^{**}} = \frac{n}{s_w} \qquad \ldots(11)$$

$$k^{**} > k^*, \text{ the root of Pasinetti's } f'(k^*) = \frac{n}{s_c}$$

$$\lim_{t\to\infty} r = r^{**} = f'(k^{**}) < f'(k^*) = r^* = \frac{n}{s_c}$$

$$\lim_{t\to\infty} \frac{\dot{K}_c}{K_c} = r^{**}s_c < r^*s_c = n, \text{ or } \lim_{t\to\infty} \frac{\dot{k}_c}{k_c} < 0.$$

To reinforce our common sense proof, suppose the system has come into the k^* configuration defined by Pasinetti's $f'(k^*) = n/s_c$. Fig. 1 shows it will pass out of that state when $s_w > \alpha(k^*)s_c$. Why? Because for any positive division of k^* between k_c and k_w we shall be at the vertical level of some definite point on the descending curve of Fig. 1. On that line k_c and k_w would grow in balance; but now take notice that s_w has increased, moving us rightward of the curve (*due eastward!*); if lower s_w would keep K_w growing in balance with the unchanged rate of growth of K_c, then higher s_w means K_w grows *faster* at k^* than does K_c. Hence, we are on our way to a new equilibrium point—the one we have called k^{**}, identifiable by considering workers as the only (that is, as the overwhelmingly only) source of asymptotic saving.

This leads to another, perhaps more sophisticated way, of understanding the two cases—Pasinetti's and its dual opposite. Consider the artificial condition where the capitalist class never gets a chance to do its Marxian " primitive accumulation ": set K_c initially zero. The resulting steady state is the familiar one of Solow *et al.*, in which the uniform s_w of the single class determines the golden age we have denoted by k^{**}, the root of $f(k^{**})/k^{**} = n/s_w$. Now, is this special steady-state *stable* when we test it by bringing

[1] F. H. Hahn and R. C. O. Matthews, op cit., devote a page of their masterly review to the Pasinetti analysis. After describing the Pasinetti regime, they correctly mention in a footnote the possibility of the Dual regime in which $k_c \to 0$; but they incorrectly state the possibility of a third regime where " the assets of wage-earners, while not growing at the same rate as those of capitalists have become a negligible fraction of the latter " (p. 799, n. 1). Actually, if s_w is positive, our general theorem shows that it is impossible for $k_w \to 0$ in the manner alleged.

an iota of positive K_c into existence? The answer is clearly Yes, if we are in the range of the dual theorem where (9) holds. For a little K_c will initially grow at the rate $s_c r^{**}$. Now, since by definition $\frac{k^{**}}{f(k^{**})} = \frac{s_w}{n}$ and $\frac{k^*}{f(k^*)} = \frac{\alpha(k^*)s_c}{n}$, (9) implies $\frac{k^{**}}{f(k^{**})} > \frac{k^*}{f(k^*)}$ and therefore $r^{**} < r^*$. Thus $s_c r^{**} < s_c r^* = n$, and K_c will grow slower than L, so that $k_c = K_c/L$ will tend to vanish again, moving us back toward the initial solution. On the other hand, if we are in the Pasinetti range, $s_w < \alpha(k^*)s_c$, then, by the same token, $s_c r^{**} > s_c r^* = n$ and hence the initial k_c will grow displacing the initial equilibrium to a new equilibrium with $r = r^*$ and hence $k = k^*$, the Pasinetti solution.

Fig. 2 illustrates the critical boundary of the Pasinetti range and of the range of the dual theorem. The heavy n/s_c line intersects the marginal-product curve $f'(k)$ at the k^* abscissa. If s_w is very small, the n/s_w line marked 1 will intersect the average product curve $f(k)/k$ at a k_1^{**} level lower than k^*. So Pasinetti's theorem will apply. Alternatively, let s_w be so large as to bring the n/w line down to 3, which intersects the AP curve at $k_3^{**} > k^*$. Then workers' saving will dominate (and ultimately completely dominate) and we are in the domain of the anti-Pasinetti dual theorem. Evidently, the critical watershed is at 2, where n/s_w intersects AP at the same k^* level where n/s_c intersects MP. The critical ratio for s_w/s_c is where it equals $(MP/AP)^*$; but this last is precisely the definition of relative capital share $\alpha(k^*)$.[1] (The aesthetic eye will resent our always writing $s_w \gtreqless \alpha(k^*)s_c$ and not its dual $s_w \gtreqless \alpha(k^{**})s_c$; but careful study of Fig. 2 will show that these are equivalent criteria.)

From equations (10) and the above discussion, we conclude that when s_w exceeds a modest critical level, Pasinetti's theorem must be replaced by the following:

Dual Theorem. When $s_w \geqq \alpha(k^*)s_c = nf(k^*)/k^*$, the steady-state growth path to which the system tends has the following characteristics:

(i) The rate of interest, the capital-labour and capital-output ratio and therefore also the distribution of income between wages and profits are completely independent of the capitalist propensity to save s_c.

(ii) The average-product-of-capital (and its reciprocal the capital-output ratio) are independent even of the form of the production function, depending only on the rate of growth n and the workers' saving propensity according to the formula $(Y/K)^{**} = s_w/n$.

(iii) The remaining ratios and the rate of interest depend on s_w/n and on the form of the production function.

(iv) If K_c is ever positive, its ultimate growth rate will be less than that of the system as a whole.

[1] Why did Dr Pasinetti's mathematics not warn him that his was only a fraction of the story? Well, it did. But it whispered rather than shouted. In his equation (13), p. 272, he notes that the factor $(I - s_w Y)$ must not be zero if he is to be able to cancel it out and arrive at his final formulas. Who can be blamed for thinking such vanishing to be an unimportant singular case of razor's-edge width? Unfortunately, it is the content of our Dual theorem that the above expression will asymptotically vanish for all $s_w \geqq \alpha(k^*)s_c$, as will be evident if only k_w^{**} counts asymptotically there. $P/K = r^{**}$ is still determinate, but not from the p. 272 equations. Our Dual theorem corrects the absurdities implied by an attempt to confine reality to the Pasinetti theorem's consequences. Thus, suppose the last capitalist went permanently to work for a minute a day: the careful reader of pp. 272-274 might be forgiven for concluding that economic indeterminacy would suddenly result, as he ponders over the words . . . " the behavioural relation $(s_c P_c)$ determining the rate of profit drops out of the picture altogether and the rate of profit becomes indeterminate. . . . (The parameter s_w, which remains cannot determine the rate of profit!) " (p. 274). Once our Dual theorem is understood, no cataclysmic indeterminacy sets in: the determinate asymptote for P/K becomes $r^{**} = \alpha(k^{**})n/s_w$, where k^{**} is defined above in terms of n/s_w alone. Economic intuition is vindicated.

In Dr Pasinetti's important footnote on p. 216, the differential equation $s_c[F(K, L) - W] = knL + L\dot{k}$ is erroneous, being patently inconsistent with our (3), (5) and later (5″); an asymptotic identity was wrongly used in deriving this differential equation purporting to be valid at all times. The correct formula, which together with (3) or (5) will lead to all valid cases, is given by our equation (5″).

The general formulae that cover both theorems can be stated as follows. For any variable X, let X^{∞} denote the limit which that variable approaches as $t \to \infty$.

General Theorem. Then: (i)

$$k^{\infty} = \text{Max}\,(k^*, k^{**}),\ \text{where} \begin{cases} k^* = \text{root of } f'(k^*) = \dfrac{n}{s_c} \\ k^{**} = \text{root of } \dfrac{f(k^{**})}{k^{**}} = \dfrac{n}{s_w} \end{cases} \qquad \ldots(12)$$

$$r^{\infty} = \text{Min}\,(r^*, r^{**}) = \text{Min}\,[f'(k^*), f'(k^{**})] = \text{Min}\left[\left(\frac{s_c}{n}, f'(k^{**})\right)\right]$$

$$(Y/K)^{\infty} = \text{Min}\,[A(r^*), A(r^{**})] = \text{Min}\left[A(r^*), \frac{s_w}{n}\right].$$

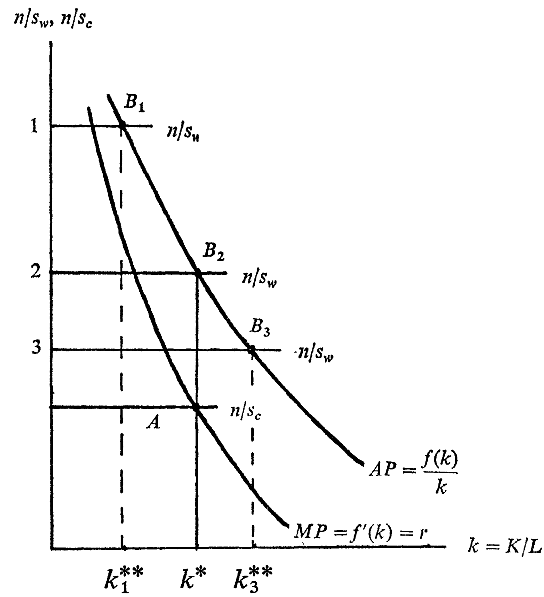

FIGURE 2

The intersection of n/s_c with the MP curve of interest at A determines the Pasinetti equilibrium. It is the actual asymptote for the system if, and only if, s_w is small enough to make the n/s_w intersection with the average product curve occur to the right of A, as at B_3. If s_w exceeds the product of s_c and capitalists' relative share α_k, the k_3^{**} Anti-Pasinetti point becomes the asymptotic equilibrium.

(ii) The ultimate allocation of k^{∞} between $(k_c^{\infty}, k_w^{\infty})$ is given by $(0, k^{**})$ = (0, function of s_w/n alone) if $s_w \geqq \alpha(k^*)s_c$; and, if $s_w < \alpha(k^*)s_c$, by positive numbers $(\beta_1 k^*, \beta_2 k^*)$, where k^* depends only on s_c/n and where the fractions β_j depend (for fixed k^*) only on s_w.[1]

[1] The reader can construct a diagram dual to Fig. 1. Now s_c is on the horizontal axis, with s_w fixed. On the vertical axis, k^{∞} is a horizontal line marked k^{**} and k_w^{**} up to the critical $s_c = s_w/\alpha(k^*)$ level. In this same dual range $k_c^{\infty} = k_c^{**} = 0$, the horizontal axis. For larger s_c, we are in the Pasinetti range: k^{∞} now rises in a convex-from-below broad market k^*; k_c^{∞} rises from zero to positive levels along the k_c^* branch, but could eventually decline with s_c if diminishing returns were strong enough; k_w^{∞}, on the other hand, must be helped by an increase in s_c, just as it is by an increase in s_w.

VI. STABILITY ANALYSIS

Now we revert back to the neoclassical model to show that our golden-age equilibria, (k_c^*, k_w^*) or $(0, k^{**})$, are stable in the sense that when the s_w/s_c is appropriate for each it will in fact be approached asymptotically from all sufficiently nearby values (k_c, k_w). We use standard *local* stability analysis to prove that any small disturbance from equilibrium will be followed by an asymptotic return to it. We are unable to state a global stability theorem in the Pasinetti case because we are not able to rule out the possibility of limit cycles. If they could be ruled out, our unqualified local stability conditions would entail global stability for all positive K's.

Begin with Pasinetti's (k_c^*, k_w^*). Around this positive level, expand the differential equations (5) in a Taylor's series, retaining only the first-degree terms in the divergences $(k_c - k_c^*, k_w - k_w^*) = (x_1, x_2)$, deriving *linear* differential equations for (x_1, x_2). For notational convenience, replace the subscripts c and w by 1 and 2, to get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \dot{k}_i}{\partial k_j} \end{bmatrix}^* \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \ldots(13)$$

where the * outside the matrix indicates that we evaluate its elements as the partial derivatives of (5) at the equilibrium levels (k_c^*, k_w^*). [In a moment, we shall perform similar stability analysis of our anti-Pasinetti equilibrium $(0, k^{**})$, putting ** on the related matrix of partial derivatives.]

The calculated result is

$$\begin{bmatrix} \dfrac{\partial \dot{k}_i}{\partial k_j} \end{bmatrix}^* = \begin{bmatrix} s_c f'' k_c & s_c f'' k_c \\ -s_w f'' k_c & -s_w f'' k_c + (s_w f' - n) \end{bmatrix}. \qquad \ldots(14)$$

Putting $-\lambda$ in the diagonal of the matrix, we calculate its characteristic polynomial

$$\begin{aligned} \Delta(\lambda) &= \lambda^2 + [(s_c - s_w)(-f'')k_c + (-s_w f' + n)]\lambda + [(-s_w f' + n)(-f'')ks_c] \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2) = 0. \end{aligned} \qquad \ldots(15)$$

The bracketed coefficients of this polynomial are seen to be both positive, because every factor in every term is assuredly positive for $s_w < \alpha(k^*)s_c < s_c$, $f'' < 0$, $s_w f' < s_c f' = n$. And the positiveness of these coefficients are the known necessary and sufficient conditions that the real parts of both λ roots be negative—which does assure the local stability property.

How local is local? That is never easy to specify. However, if initially (or, equivalently, ever) $K_c = 0$, obviously it is forever zero and the motion will approach the $(0, k^{**})$ asymptote. However, the slightest positive perturbation of k_c will send the system away from that point so long as $s_w < \alpha(k^*)s_c$. And our analysis confirms that there exists no locally unstable equilibrium point in the positive quadrant.

Will the ultimate approach to (k_c^*, k_w^*) be by means of damped oscillations or be monotonic? That depends on whether the λ_i roots are complex or real. Either case is possible. Thus rewrite the characteristic polynomial in the self-defining notation

$$\lambda^2 + [\alpha_1\alpha_2 + \alpha_3]\lambda + \alpha_3\alpha_2 = 0.$$

Then by making the discriminant $(\alpha_1\alpha_2 + \alpha_3)^2 - 4\alpha_3\alpha_2$ negative, we get damped oscillations of the focus type; and by making it positive, we get monotonic stability of the nodal type. The numerical cases $(\alpha_1, \alpha_2, \alpha_3) = (1, \frac{1}{2}, 1)$ or $(1, \frac{1}{2}, 0{\cdot}01)$, both of which appear to be admissible, lead respectively to damped oscillations and simple decay.

Let us now test the local stability of our dual equilibrium $(0, k^{**})$, which prevails when $s_w > \alpha(k^*)s_c$. Now our linear approximation has the matrix

$$\begin{bmatrix} \dfrac{\partial \dot{k}_i}{\partial k_j} \end{bmatrix}^{**} = \begin{bmatrix} s_c f'(k^{**}) - n & 0 \\ 0 & s_w f'(k^{**}) - n \end{bmatrix} \qquad \ldots(16)$$

whose λ roots will both be clearly negative because of the relations

$$0 = s_w \frac{f(k^{**})}{k^{**}} - n > s_c f'(k^*) - n > j_w s_c f'(k^{**}) - n$$
$$0 = s_w \frac{f(k^{**})}{k^{**}} - n > s_w f'(k^{**}) - n. \qquad \ldots(17)$$

With local stability assured, what about global stability? A phase diagram can verify that our dual equilibrium is stable no matter what non-negative values (k_c, k_w) are perturbed to take on initially. For now the locally stable point $(k_w^{**}, k_c^{**}) = (k_w^{**}, 0)$ falls on the horizontal axis, so that for a limit cycle to surround it the variables would have to become negative, which is a contradiction. Thus, our dual-theorem equilibrium has true stability in the large.

It is remarkable that this general system has both a unique equilibrium and unconditional local stability. To emphasize this recall that a similar system in which all profits are saved at an s_c rate and all wages at an s_w rate can, even in the well-behaved neoclassical case, easily (for $s_w > s_c$) have multiple equilibria (of which some are unstable).[1]

VII. CONCEPT OF A GENERALIZED GOLDEN AGE

This is perhaps the place to dispose of a terminological ambiguity. A person unsophisticated in the conventions of mathematics might be inclined at first to expect *everything* in a " golden age " to be growing at *exactly* the same percentage rate. And in the case where one of the variables, say K_c, were zero, he might be puzzled over how to interpret its $\dot{K}_c/K_c = 0/0$ rate of growth. The applied mathematician is used to singular cases, where nice distinctions must be made between things being non-negative and positive. To avoid sterile controversy over semantics, he will define a " generalized golden age " as a state which, including the standard one as a special case, goes on to include steady-state configurations in which some of the equilibrium ratios are zero.

Our dual-theorem case is an example in point. K_c, if once positive, grows forever, never approaching zero. But the ratio $k_c \to 0$ because L grows faster than K_c. Now consider two situations, one where $K_c^1 \equiv 0$ and $k_c^1 \equiv 0$, and two where $K_c^2 > 0$ and $k_c^2 \to 0$. Although $k_c^2 \to k_c^1$, the divergence between K_c^2 and $K_c^1 \to \infty$! The mathematician, with his concepts of " relative stability " of ratios like $(K_c/K_w, K_c/K, K_c/L)$ is not at all perturbed that some different extensive variables show infinitely-divergent behaviour. And actually, once the literary economist has thought the matter through, he should not be perturbed either. For the phenomenon has already been occurring unnoticed in the standard Pasinetti case and is not peculiar to the dual-theorem case.

To see this let $s_w < \alpha(k^*) s_c$, and suppose the system starts out on the equilibrium path, with $[K_c(t), K_w(t), K(t)] = [K_c^*(t), K_w^*(t), K^*(t)] \equiv [k_c^* e^{nt}, k_w^* e^{nt}, k^* e^{nt}]$. Now suppose that at some date t_0 we bomb out of existence some capital, say some K_c. After t_0 the system follows a path $[K_c^1(t), K_w^1(t), K_c^1(t)]$ different from the equilibrium path, because of the initial disturbance $K_c^*(t_0) - K_c^1(t_0) > 0$. As we know from the stability analysis, all divergences of *per capita* magnitudes from equilibrium must go to zero in the limit: $(k_c^1, k_w^1) \to (k_c^*, k_w^*)$. But does this " relative stability " also imply " absolute stability ", namely that $| K_c^1(t) - K_c^*(t) |$ and $| K_w^1(t) - K_w^*(t) |$ tend to zero? The answer is, not necessarily, surprising as this may seem to the literary intutition. It is easy to show by numerical illustrations and by generalized stability analysis that the absolute divergence,

$$| K_c^1(t) - K_c^*(t) | = \delta(t)$$

[1] See the papers by Uzawa, *et al.* of the Hahn-Matthews bibliography cited in footnote 2, p. 270. Mr Edwin Burmeister, in a current MIT Ph.D. thesis, has contributed to this same subject. We leave as an open research question the problem of whether the Pasinetti equilibrium can fail to be globally stable because of the existence of a limit cycle.

need not approach zero, but may instead *increase without bound*; in fact when plotted on semi-log paper (as in Fig. 3*a*) the absolute discrepancy may approach asymptotically a *positively* sloped straight line with a slope smaller than n, the system's natural rate of growth. In other words the discrepancy can grow exponentially, though at a rate smaller than n.[1]

Thus, we have already had in the Pasinetti range the same infinite divergence of absolute magnitudes that is apparent in the Dual range. Why not? When on *semi-log paper* one curve approaches a positive-sloped straight line asymptotically, the eye sees a vanishing divergence; but when we translate into absolute numbers the magnitudes of the divergence, it can often be shown to become infinite. The sophisticated eye knows how one must allow for the properties of exponentials and ratios.

Fig. 3*a* shows how $(K_c, K_w, K) \rightarrow (k_c^* L, k_w^* L, k^* L)$ in a Pasinetti case, as indicated by the fact that all their curves end up paralleling L's growth rate. The broken lines show that no lasting *per capita* improvement will result for society from a small increase in s_w alone: the new K path approaches the old asymptote, as K_c comes to lose what K_w gains.

At the bottom of Fig. 3*a* we have plotted the absolute divergence from the equilibrium path discussed above, namely $\delta(t) = K_c^*(t) - K_c^1(t)$. Note that $\delta(t) \rightarrow$ a straight-line growth path with slope $< n$.

Fig. 3*b* shows the asymptotic behaviour in our dual case. K_w approaches the curve of total K, which ends up parallel to the growth of L. But K_c ends up with the slower growth rate $s_c f'(k^{**}) < n$, and becomes of vanishing relative importance in the limit. Now if s_w is increased a little more, the broken-lines show that K and K_w end up *permanently* higher, with *per capita* product permanently higher even though ultimate growth rate is the same. The capitalists end up permanently worse off, with an ultimate growth rate that is definitely lower. [The reader can show that an increase in s_c alone would, in this case of 3*b*, raise capitalists' ultimate growth rate but have a negligible ultimate effect on society's growth rate or *per capita* magnitudes. In Fig. 3*a*, the reader can show that an increase in s_c will permanently raise the *per capita* ratios k^* and k_w^*, but k_c^* can actually be lowered if the elasticity of substitution is low enough.] [2]

VIII. GENERALIZATION TO MANY CAPITALIST OR WORKER CLASSES

Now that we have given a rigorous analysis of the Pasinetti and Dual theorems, we can rapidly consider various generalizations. Dr Pasinetti indicated in his paper that his theorem could be applied to a world in which " the non-capitalists might be divided into any number of sub-categories one likes " (p. 274), each having a different saving propensity, say $s_w^1 > s_w^2 > \ldots > s_w^M$, provided, of course, $s_w^1 < \alpha(k^*) s_c$. He has not, however, investigated the case in detail nor the case where there are also any number of subcategories of capitalists, having different saving propensities $s_c^1 > s_c^2 > \ldots > s_c^N$. Clearly, the simple formula $r = n/s_c$ cannot then hold, since it would give N different (and inconsistent) values for r. The correct solution is found by examining the following extension of (5)'s saving equations, where k is short for $k_c^1 + \ldots + k_c^N + k_w^1 + \ldots + k_w^M$, and $\lambda_j = L_j/L$, the constant relative fraction

1 Mathematically, previous stability analysis showed that $k_c(t) = k_c^* + a_1 \exp(\lambda_1 t) + a_2 \exp(\lambda_2 t) + R$, where the remainder term R is dominated by exp [t *max* (real part of λ_1, real part of λ_2)] = exp (tm), for short. Perturbations affect only the a_i coefficients. Hence, the divergence defined above can be written as

$$\delta(t) = b e^{(n+m)t}[1 + \rho(t)]$$

where $\rho(t) \rightarrow \rho(\infty) = 0$. Provided $n > -m = -\text{Max}$ [real part of λ_i], we have infinite absolute divergence. One instance suffices to show that this condition is easily possible. Let $f(k) = \alpha^{-1} k^{\alpha}$, $s_w = 0$ (or " small "), $s_c > \alpha s_w$. Then, we find from (15) above, $m = \text{Max}\,[-n,\ s_c f'' k_c] = \text{Max}\,[-n,\ -n(1-\alpha)] = -n(1-\alpha)$. Hence, $n > (1-\alpha)n = -m$ because $1-\alpha$ must be a positive fraction. Q.E.D.

2 At this point, the interested reader can be referred to the Appendix, which brings to completion the above analysis by considering explicitly the problems that arise when K/L tends either to zero or infinity.

of labourers who belong permanently in the jth non-capitalist category:

$$\dot{k}_c^i = [s_c^i f'(k) - n]k_c^i \qquad (i = 1, 2, \ldots, N)$$

$$\dot{k}_w^j = s_w^j[f(k) - kf'(k)]\lambda_j + [s_w^j f'(k) - n]k_w^j \quad (j = 1, \ldots, M) \qquad \ldots(18)$$

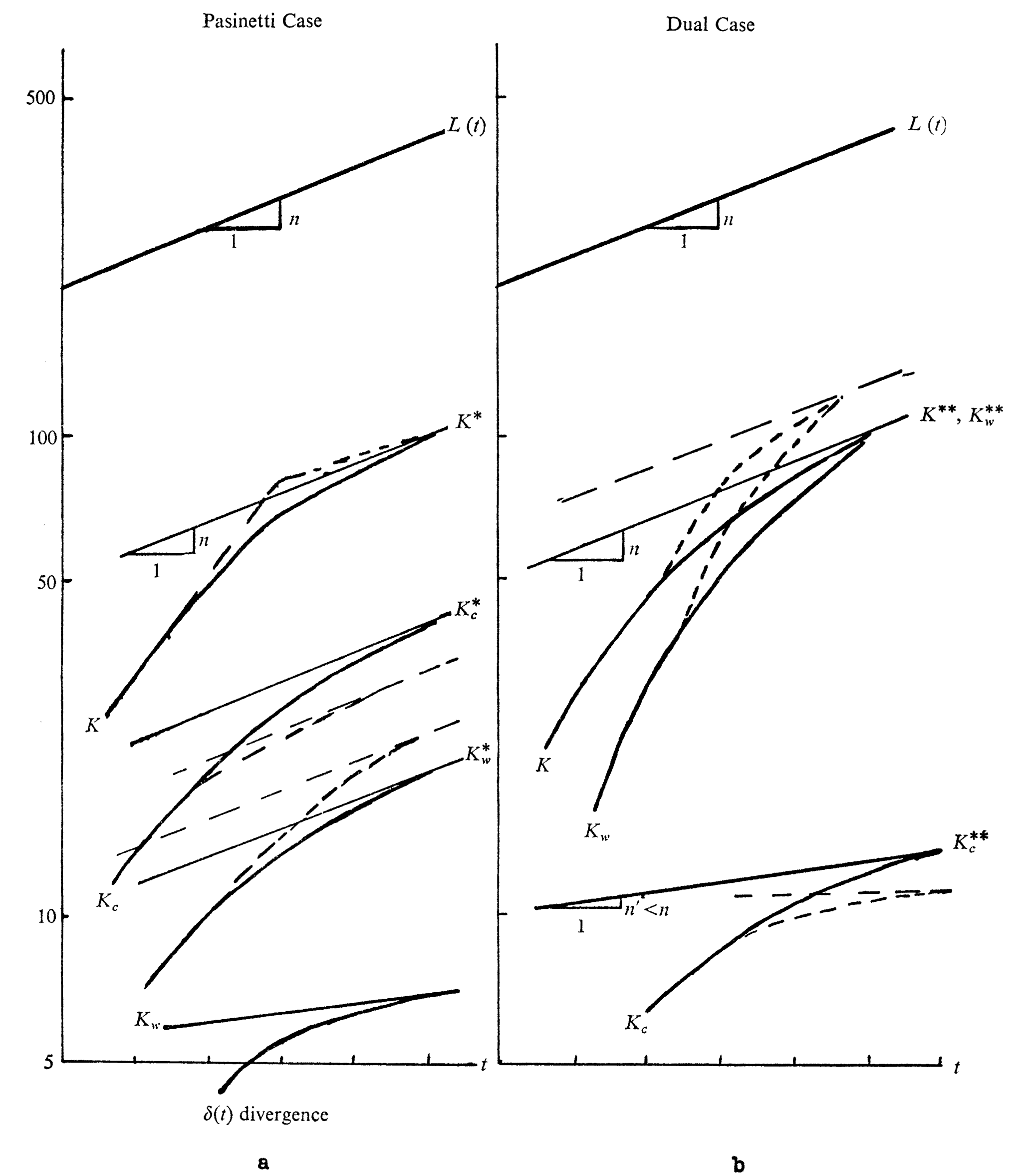

FIGURE 3

In the Pasinetti regime, K, K_c and K_w all approach the asymptotic growth rate of labour. The broken lines in 3a show the effects of a small increase in s_w, with ultimate K trend unchanged, as K_c loses what K_w gains. $\delta(t)$ plots the absolute divergence from the equilibrium path attributable to an initial reduction in K_c, a divergence that goes to infinity, but becomes negligible relative to K or L.

In the Dual regime of 3b, K and K_w approach the growth rate of L; K_c grows toward infinity, but at an asymptotic rate lower than K_w and K, so that the generalized golden age is dominated by K_w.

Quite obviously setting $(k_c^1, \ldots, k_c^N)$ simultaneously equal to zero will lead to a self-contradiction unless

$$s_c^1 f'(k^*) - n = 0$$
$$0 = k_c^2 = \ldots = k_c^N.$$

Inevitably K_c^1 will grow faster than K_c^i; at best, k_c^1 can become a positive constant with all other k_c^i diminishing to zero, since $\dot{K}_c^i/K_c^i < \dot{K}_c^1/K_c^1$.

There are then only two possibilities for

$$[k_c^1(\infty), k_c^2(\infty), \ldots, k_c^N(\infty); k_w^1(\infty), \ldots, k_w^M(\infty)],$$

namely

$$[a_1^*, 0, \ldots, 0; b_1^*, \ldots, b_M^*]$$

where the positive b_j^* and a_1^* add up to k^*, the Pasinetti root of $f'(k^*) = n/s_c$; or

$$[0, 0, \ldots, 0; b_1^{**}, \ldots, b_n^{**}]$$

where $\Sigma b_j^{**} > k^*$. These Pasinetti and Dual-Theorem extensions can be summarized in a third general theorem that includes all previous theorems as special cases.

General Theorem. Let k^* be the root of $f'(k^*) \doteq n/\text{Max}_i(s_c^i)$ and designate the respective extended Pasinetti and extended Dual cases according to whether

$$s_w^1 = \text{Max}[s_w^1, \ldots, s_w^M] \lesseqgtr \alpha(k^*)\, \text{Max}[s_c^1, \ldots, s_c^N]. \qquad \ldots(19)$$

(i) In the extended Pasinetti case, all

$$(k_c^i;\ k_w^j) \to (a_1^*, 0, \ldots, 0;\ b_1^*, \ldots, b_M^*), \text{ with}$$

$$f'(a_1^* + b_1^* + \ldots + b_M^*) = r^* = \frac{n}{s_c^1}. \qquad \ldots(20)$$

While the interest rate is independent of the form of the production function, the ultimate values for other starred equilibrium magnitudes depend on n/s_c^1 according to the form of the production function. [See Equation 22 below.]

(ii) In the extended dual case $(k_c^i;\ k_w^j) \to (0, \ldots, 0;\ b_1^{**}, \ldots, b_M^{**})$, where

$$k^{**} = b_1^{**} + \ldots + b_M^{**}$$

exceeds k^*, and is the root of

$$\sum_{j=1}^{M} \frac{[f(k^{**}) - k^{**}f'(k^{**})]s_w^j \lambda_j}{n - s_w^j f'(k^{**})} - k^{**} = 0. \qquad \ldots(21)$$

Now r^{**} is below r^* and, like all equilibrium double-starred magnitudes is a function of n, all the (s_w^j) and (λ_j), and depends on the form of the production function.

(iii) In every case, the relative magnitudes of b_j^* or b_j^{**} are increasing functions of the respective s_w^j: the workers' group with the highest saving propensity ends up with the largest capital *per capita*, which in the extended Pasinetti case is completely at the expense of the other groups in society.

(iv) There is a fundamental asymmetry between a worker and capitalist category. Thus, if one of two capitalists with the same s_c^1 starts out with more wealth, he " ends up " permanently ahead by a finite " *per capita* " amount of k_c^*. Indeed " his " extra k_c^* forces an equivalent reduction in the k_c^* of his class colleagues, since the *total* k_c^* of the class is unchanged and hence the k_w^{j*} of all workers is left unchanged. This last conclusion holds even in the dual case, except that of course *all* k_c^* end up zero in that case.

However, suppose some one worker or subgroup of workers should somehow begin with more k_w^j (or more k_w^j/λ_j), the ultimate *per capita* effect upon other workers in the same *j*th category, upon other workers in a different category, or on the ultimate k_c^∞ of any capitalist group is null in *every* case. High initial k_w^j/λ_j gets washed out in the end since always equation (18) ends up with

$$\frac{k_w^j(\infty)}{\lambda_j} = \frac{f(k^\infty)-k^\infty f'(k^\infty)}{\dfrac{n}{s_w^j} - f'(k^\infty)}$$

$$k_c^1(\infty) = k^\infty - \sum_{j=1}^{n} k_w^j(\infty) \qquad \ldots(22)$$

independently of any initial or penultimate conditions.

[As in our earlier Section VII discussion, the effects on absolute (K_c, K_w, K) magnitudes must be distinguished from effects on relative (k_c, k_w, k) magnitudes. If one member of the s_c^1 capitalist class has some of " his " K_c^1 destroyed by fire, in enough time his absolute wealth will have fallen infinitely behind that of his colleagues, even though his (diminished) principal grows at the same rate as theirs and though his share of $k_c^1(\infty)$ declines by only a finite amount. Suppose a cohort of workers lose some K_w^j to fire. We have seen that time washes out this effect as far as their ultimate *per capita* $k_w^j(\infty)/\lambda_j$ is concerned. But the absolute effect of the fire on their $K_w^j(t)$, which is given by the $\delta(t)$ divergence of $k_w^j(t)$ exp (nt), can be shown by an extension of footnote 11*c*'s mathematics to be capable of growing like $\exp(n+m)t$ where $n+m>0$, and m is the maximum real part of the λ roots appropriate to the linear stability analysis of (18).]

The proof of all these statements comes from putting all the $\dot{k}$ terms on the left-hand side of (18) equal to zero, and then searching the resulting statical relations for relevant non-negative solutions. Stability analysis is not presented here, but would represent a straightforward extension of our earlier discussion.

IX. SOME IMPLICATIONS FOR THE GOLDEN RULE AND ITS ALLEGED RELATION TO SOCIALISM

Our General Theorem has one interesting corollary that Dr Pasinetti has commented on, and somewhat surprisingly linked with a " socialist system ". (Section VIII, pp. 277-278.)

Corollary. If there is one capitalist who *permanently* succeeds in saving *all* his income (and if Max $[s_w^j] < \alpha(k^*)$ where k^* is the root of $f'(k^*) = n$), then in enough centuries the system will—independently of all other saving propensities s_w^j and s_c^i—approach the Golden-Rule state of Swan-Phelps, that golden age characterized by maximum *per capita* consumption, in which $r^* = n$ and $I = P$.

Dr Pasinetti has associated this case of $s_c^1 = 1$ with socialism on the ground that " the state as such cannot consume " (p. 277) and hence must have a propensity to save equal to unity. However, since the government uses resources to provide many kinds of current services which it could finance with its property income in lieu of taxes and since it could always distribute some of its property income by gratuitous transfers, it would seem that socialism, in the usual sense of the term as involving public *ownership* and *management* of (some or all) means of production, is neither sufficient nor necessary for the analytic result enunciated. This is of course not to deny that government policy can be an important determinant of the rate accumulation in socialist as well as mixed economies.

We emphasize the centuries that may be involved to stress that we are talking here and everywhere of hypothetical steady-states which will *never* quite be reached from other states and which may be closely approximated only after such long periods of time as to make the models' realism questionable. Of course, this is no more a point against Dr. Pasinetti than against Solow, Harrod, Joan Robinson, Meade, Samuelson-Modigliani and other golden-age mongers.

X. NEOCLASSICAL DISTRIBUTION THEORY ABANDONED OR GENERALIZED

Most of the above seem to rest on marginal productivity notions of the Clark-Wicksteed-Solow-Meade type. But if one examines the basic Equations (3) and all the steps leading up to (5) and beyond,[1] it will be found that no direct use has been made of equation (2)'s marginal productivity relations. There is no necessity to identify the interest rate relations $r = f'(k)$ with the partial derivative symbol $\partial F(k, 1)/\partial k$ or with $df(k)/dk$! All we need is that r should be a determinate function of K/L, but that function need not be the above derivative. Thus, even if there are not smooth substitutability properties posited for (1)'s production function or even if Chamberlinian imperfect competition intervenes in factor or commodity markets, our analysis can still be applied. If Kalecki, or Boulding, or Hahn, or Kaldor, or Schneider, or Walter Reuther, or Thünen come forward with some alternative theory of distribution, provided only that the profit rate is a declining function of the ratio of capital to labour—call it $r = \phi(K/L)$ [2]—both the Pasinetti formalism and our various duals and generalizations of them remain valid.[3]

Now the positive wage rate becomes $f(k)-k\phi(k)$ rather than $f(k)-kf'(k)$ and $\alpha(k)$ becomes $k\phi(k)/f(k)$ rather than $kf'(k)/f(k)$. The basic equations are now

$$\dot{k}_c = [s_c\phi(k)-n]k_c$$
$$\dot{k}_w = s_w[f(k)-k\phi(k)]+(s_w\phi(k)-n)k_w.$$

As before, we are in the Pasinetti or Dual anti-Pasinetti range depending upon whether

$$k^{**} < k^* \text{ or } k^{**} > k^*$$

where

$$k^* \text{ is root of } \phi(k^*) = \frac{n}{s_c}$$

$$k^{**} \text{ is root of } \frac{f(k^{**})}{k^{**}} = \frac{n}{s_w}.$$

[1] The stability matrix (14) and (16) is a minor exception to this. See footnote 1, p. 289 for modifications in our stability analysis when marginal productivity relations are dropped. The Appendix discussion of pathology would also require some minor modifications.

[2] For an excellent survey of these diverse theories see K. W. Rothschild, " Some Recent Contributions to a Macro-economic Theory of Income Distribution," *Scottish Journal of Political Economy*, **8** (1961), 173-179. [Warning: many macroeconomic theories, e.g. Kaldor's, exclude the existence of a $\phi(k)$ function].

[3] After we had completed this analysis, our attention was called to the valuable paper, J. E. Meade, " The Rate of Profit in a Growing Economy ", *Economic Journal*, **73** (1963), 665-674, which gives many of the results for the case of a Cobb-Douglas function that we had also arrived at. While we find much to admire in this paper, we think unfortunate the impression that many readers will get from Professor Meade's words " . . . the ' neo-classical ' result will be true in its simplest form if $s_w > \alpha(k^*)s_c$ [our notation] while the ' neo-Keynesian ' will be true in its simplest form if $s_w < \alpha(k^*)s_c$" (p. 669). If the term " neo-classical " is used as we use it, merely to indicate existence of smooth derivatives $\partial F/\partial K$, $\partial^2 F/\partial K^2$, and competitive imputation of factor prices, then the range of validity of the Pasinetti theorem as against our Dual (and General theorems) has nothing peculiarly to do with this smooth differentiability issue—as our earlier analysis showed and as this section further illuminates. This remark of ours would seem to apply too to Dr Pasinetti's " Comments " on Meade's paper, *Economic Journal*, **74** (1964), 488-489; but it should be interpreted in the context of our later Section 12's discussion. If neo-Keynesian is used merely as an O.K. word for valid arithmetic identities—such as the implications of $\dot{Y}/Y = \dot{K}/K = \dot{L}/L = n$—we are all neo-Keynesians. Along with rejecting the notion that some particular range of the s_w/s_c parameters has a neo-Keynesian as against neoclassical significance, we question the imperialistic notion that the " neo-Keynesian results hold in general " if the words " neo-Keynesian " are here interpreted to involve any or all of the following notions: *I* must be thought of as in some sense autonomous; marginalism is a modern irrelevancy; effective demand problems are always vital; income shares alter (in the long or short run) to equilibrate full employment; causation of interest or profit determination runs from growth rates of labour and not from impatience, thirftiness, and " technical productivity ". Many of these elements enter into certain models that we often choose to analyse, but there is nothing universal about such behaviour equations.

This presupposes that more capital relative to labour means lower r, so that $\phi'(k)<0$ even if $\phi'(k) \neq f''(k)$. We shall again have as watershed between the two regimes

$$s_w<\alpha(k^*)s_c \quad \text{Pasinetti}$$
$$s_w>\alpha(k^*)s_c \quad \text{Dual.}$$

Fig. 2 still applies, but now with the *MP* curve no longer having a marginal productivity interpretation. The *AP* curve still does have an average product interpretation.

Let us illustrate by a " bargaining power or a just-wage or Kalecki markup theory of distribution ". A stringent example will be the case of fixed-coefficients of production à la Leontief and early Walras. Suspending our disbelief in the realism of such a model, we write

$$C+\dot{K} = \text{Min}\left(\frac{K}{k^\dagger}, \frac{L}{1}\right) = Lf(k) = L\,\text{Min}\left(\frac{k}{k^\dagger}, 1\right) \qquad \ldots(1')$$

where L has been defined in units such that 1 L is needed to produce 1 unit of C or of $\dot{K}$, and where $k^\dagger$ is the minimal capital-output ratio (measured in years). In this special neo-neoclassical model, capital and labour are " needed " in fixed proportion, and whichever happens to exceed this critical balance is redundant from a current technical point of view. (I.e., $f'(k) = d\,\text{Min}\,(k/k^\dagger, 1)/dk = 1/k^\dagger$ for $k<k^\dagger$; $= 0$ for $k>k^\dagger$; and is indeterminately defined as anywhere inbetween for $k = k^\dagger$.) Marginal productivity can certainly not determine $\alpha(k^\dagger)$ because of the kinked corner that $f(k)$ has at $k = k^\dagger$.

Jettisoning marginal productivity, suppose collective bargaining, the State, or Kalecki-Chamberlin markup always gives wages three-fourths the total product, giving the remaining one-fourth as profit or interest to owners of capital (in proportion to the quantity of capital they own). This is a fairly bizarre theory since, for all but a razor's edge of $K/L = k^\dagger$, one or the other of K and L is quite redundant and would become a free good under any regime of ruthless competitive price flexibility. (Thus, it is here implied that, even if total K were redundant, a small private owner who brings into existence some further K is able to command the same r profit as existing capital. There is then a great divergence between private pecuniary return from K and " the true social return "). The constancy of relative shares sounds, in one aspect, like a pseudo Cobb-Douglas function with coefficients $(\frac{3}{4}, \frac{1}{4})$. But why bother with Cobb-Douglas terminology? Instead, define r as

$$r = \phi(k) = \frac{\frac{1}{4}Y}{K} = \tfrac{1}{4}A(k) = \tfrac{1}{4}\,\frac{\text{Min}\,(k/k^\dagger, 1)}{k} = \tfrac{1}{4}\,\text{Min}\left(\frac{1}{k^\dagger}, \frac{1}{k}\right) \qquad \ldots(23)$$

and just carry on. In this example, both $\phi(k)$ and $A(k)$ have finite maxima:

$$\text{Max}\,\phi(k) \equiv r_m = (\tfrac{1}{4})(1/k^\dagger),\ \text{Max}\,A(k) \equiv A_m = 1/k^\dagger.$$

It is shown in the Appendix that under these conditions, if n is sufficiently large relative to s_c and s_w, then accumulation will prove insufficient to keep up with the growth of labour (in efficiency units) and $k \to 0$. The specific condition for this to happen is that

$$n>\text{Max}\,[s_c r_m, s_w A_m] = \text{Max}\,[\tfrac{1}{4}s_c, s_w]\,\frac{1}{k^\dagger}. \qquad \ldots(24)$$

When this inequality holds, *labour* will eventually tend to become technically redundant, $(k, y, w) \to (0, 0, 0)$, while $r \to r_m$. If, on the other hand, the above inequality is reversed, then *capital* will eventually become technically redundant, although r, not being related to capital's marginal productivity, will not tend to zero. We have two possible cases here:

(i) The Pasinetti Case, where $s_w<\frac{1}{4}s_c$. Here $k^\infty = k^*$, the root of

$$\phi(k^*) = \tfrac{1}{4}\,\text{Min}\left(\frac{1}{k^\dagger}, \frac{1}{k^*}\right) = \frac{n}{s_c},$$

which is satisfied by $k^* = \frac{1}{4}\frac{s_c}{n} > k^\dagger$ in view of (24). Also from (6),

$$k_w^\infty = k_w^* = \tfrac{3}{4}\frac{1}{\dfrac{n}{s_w} - \dfrac{n}{s_c}} > 0, \quad k_c^* = \frac{\dfrac{\frac{1}{4}s_c}{s_w} - 1}{\dfrac{n}{s_w} - \dfrac{n}{s_c}} > 0$$

$(y^\infty, w^\infty) = (1, \frac{3}{4})$.

(ii) The Dual Case, where $s_w \geqq \frac{1}{4}s_c$. Here $k^\infty = k^{**}$, the root of

$$A(k^{**}) = \text{Min}\left(\frac{1}{k^\dagger}, \frac{1}{k^{**}}\right) = \frac{n}{s_w},$$

which is satisfied by $k^{**} = \frac{s_w}{n} > k^\dagger$ (by 24). Also,

$$k_c^\infty = k_c^{**} = 0; \; k_w^\infty = k_w^{**} = k^{**} = \frac{s_w}{n}$$

$$r^\infty = r^{**} = \phi(k^{**}) = \tfrac{1}{4}\frac{n}{s_w} \leqq \frac{n}{s_c} = r^*;$$

$(y^\infty, w^\infty) = (1, \frac{3}{4})$.

There remains the hairline case:

$$n = \text{Max}\,[s_c r_m, s_w A_m] = \text{Max}\,[\tfrac{1}{4}s_c, s_w]\frac{1}{k^\dagger}.$$

Here we have again $r = r^*$ if $s_w < \frac{1}{4}s_c$, and $r = r^{**}$ otherwise. However, k can end up anywhere in the interval $(0, k^\dagger)$. If k ends up smaller than $k^\dagger$, labour is again technically redundant and $(y, w) \to (0, 0)$; while if $k = k^\dagger$, neither capital nor labour are redundant and $(y, w) \to (1, \frac{3}{4})$.

The above illustration, in which distribution of income was fully specified from the outset, should help to bring home clearly the basic fact that Pasinetti's theorem and our general theorem about golden age identities have nothing to do *per se* with any " alternative theory of income distribution ".

In general, however, there is no need to postulate constancy of shares as in the above example. Let, in a general model, the $\alpha(k)$ share be specified as some determinate function of factor proportions $k = K/L$, say $\alpha(k)$. Then $r = \alpha(k)F(k, 1)/k = \phi(k)$ gives the profit rate.

Recall that one of Kalecki's two theories about profits makes $\alpha(k)$ depend on the " average degree of monopoly or imperfection of competition " in the economy. Subject to the above stipulations, this is an admissable determinant of our $r = \phi(k)$ function. Or consider a theory of the " just wage ". On Thünen's gravestone was the formula for the " natural wage ", which in our notation becomes $[f(k)\partial F/\partial L]^{\frac{1}{2}} = T(k)$. Hence, $(Y - LT)/K = [f(k) - T(k)]/k$ will serve to define our $\phi(k)$.[1]

XI. A FURTHER NOTE ON FIXED COEFFICIENTS AND KALDORIAN DISTRIBUTION THEORIES

In the last section we have shown that, even if one discards competitive theory involving smooth neoclassical production functions, any model in which the profit rate r is a declining

[1] It will conduce toward stability if $\phi'(k) < 0$ in the diminishing returns case. If $f'(k) - \phi(k) = h(k)$ our differential equation (5) must everywhere have $f'(k)$ replaced by $f'(k) - h(k)$. This introduces into the stability matrixes (14) and (16), and into the coefficients of their characteristic polynomial, terms proportional to the factors $h(k)$ and $h'(k) = f''(k) - \phi'(k)$. For $|h|$ sufficiently small, stability is assured in every case. However, for large h divergences, it is possible to produce instances of instability. One conjectures, from diminishing returns considerations, that stability is always assured for $|-\phi'|$ sufficiently large. [Note: if wages and interest are to be positive and $\phi'(k) < 0$, $\alpha(k)$ must be fractional and obey the elasticity restriction: $E\alpha(k)/Ek < 1 - Ef(k)/Ek$.]

function of the K/L ratio will obey our general theorems of Pasinetti and Dual type. One such model, involving fixed coefficients, was just exhibited by way of illustration. On the other hand, macroeconomic distribution theories of the Kaldor type which provided the springboard for Pasinetti's general analysis, must now be given brief notice here.

Such models fare best under the assumption of fixed coefficients, for if smooth marginal productivities were well defined one would have no need for a genuinely alternative theory of distribution.[1] So recognizing the empirical oddity of the postulate, we posit last

[1] This sentence is worded cryptically and could be misunderstood. Marginal productivity, properly understood, provides some of the equations needed in a smooth neoclassical model for a complete theory of (factor and goods) pricing. Depending upon the time run, parameters of population growth (like n) and saving propensities (like s_c or s_w) also contribute blades of the scissors needed to determine equilibrium. The model defined by our equations (1)-(4), precisely because it possesses some simplifications of a neoclassical model (effectively one-sector, effectively one capital good, etc.), is useful to contrast with what appear to be some Cambridge attributions of causation. To test the cogency of a line of logic and the universality of an hypothesized direction of causation, it is valuable to apply them to *both* our neoclassical model *and* the fixed coefficient model of this section—and to certain intermediate cases involving a finite number of alternative activities or blueprints.

Specifically, assume s_w/s_c small, and consider $f'(k^*) = r^* = n/s_c$. Then, in our model, marginal productivity does " determine " the profit or interest rate at every instant of time, but that doesn't deny that n/s_c also does " determine " the interest rate in the long run. Both blades of the long-run scissors count. When we have two long-run unknowns—k^* and r^*—the presence of two independent equations is not an inconsistency; it is a necessity. Because the long-run " supply curve " is given by the *horizontal* level n/s_c, while the long-run demand curve is the varying function $f'(k^*)$, there is a genuine sense of long-run causation, which we share and also neoclassical writers share with Cambridge writers (like Kahn quoted below), that says: $n/s_c \rightarrow$long-run r^*.

But all of the above is quite consistent with short-run causation that runs in quite the opposite direction. In each short run, let $k(t) = K/L$ be given by past history. This provides us with a *vertical* short-run supply schedule, which intersects the $f'(k)$ schedule of demand. In the short run, for the model of (1)-(4), the " true " causation seen, in every instant of time, $k(t) \rightarrow f'[k(t)] = r(t)$. For this model, *capital scarcity relative to labour is the key to the level of the rate of interest*. Given the saving-investment propensities of (3), it is deduced that positive $[r(t)-n/s_c] \rightarrow$positive $\dot{k}$ (or, more exactly, $\dot{k}_c$ if $s_w \neq 0$) in every run of time, so that if initially $k<k^*$ and therefore $r>n/s_c$, $\dot{k}$ will be positive and k will grow, and if k ever attains k^*—the level of capital scarcity where $[f'(k^*)-n/s_c] = 0$—then $\dot{k} = 0$, and we stay in the k^* golden age. (If one wants to trace short-run effective demand sequences, the Central Bank ensures that the market interest rate, R, is put above and below the $f'(k) = r$ level, just enough to induce full-employment investment without inflationary or deflationary gaps.) The full-employment growth path of the system as determined by its differential equations is causal, in the same sense that the trajectories of the planets around the sun, following their Newtonian differential equations, are termed causal. This is a third meaning of causality in the present context, alongside the long term causal chain $n/s_c \rightarrow r^*$ and the short-run chain $k \rightarrow f'(k) = r$.

Warning: in a two-sector neoclassical model, where the sectors have different intensities and with production-possibility frontier $\dot{K} = T(K, L;\ C)$, $k(t) \rightarrow f'[k(t)] = r(t)$, would be replaced by

$$[k_c(t), k_w(t);\ s_c, s_w] \rightarrow [c, \dot{k}_c, \dot{k}_w, r = \partial T/\partial K = \partial \dot{K}/\partial K]$$

and where we would have to take into account the possibility of multiplicity of short-run equilibria (as e.g. in the anti-Pasinetti case where $s_c = s_w$ and capital goods production is sufficiently more relatively capital-intensive than consumption goods production) and of long run equilibria (as in the case just mentioned, where the ratio of the value of capital to output might *fall* while r falls).

In a general, neo-neoclassical model involving a great number of heterogeneous capital goods and processes, let $[k_1, k_2, \ldots]$ stand for a vector of per capita capital goods. Then in every short run, the " scarcity " of this " supply " of physical capital goods, taken together with the saving-investment demand-composition condition of the model, is an important determinant of the quasi-rents of all capital goods and of the spread of (own) rates of profit and interest $[r_1, r_2, \ldots]$. Thus, subject again to the possibility of multiple equilibria, $[k_1(t), k_2(t), \ldots, s_c, s_w] \rightarrow [r_1(t), r_2(t), \ldots]$. Under fixed saving propensities, steady-trend technology and population, middling-good foresight by entrepreneurs who are not in regions of liquidity-traps or profit-uncertainty traps, and certain nudges from fiscal and monetary authorities, such models often evolve into a golden age. *i.e.* $[r_1^\infty, r_2^\infty, \ldots] = [r^*, r^*, \ldots]$ where $r^* = n/s_c$; but still the general causality in each short run is from capital-good supply and demand conditions to the level and spread of profits; and in the longest run r^* is achieved only because the system has brought into being through capital-goods changes, and has kept into being through replacement and capital widening, the needed plentitude and variety and composition of the capital goods vector $[k_1^\infty, k_2^\infty, \ldots] = [k_1^*, k_2^*, \ldots]$. [See P. A. Samuelson and R. M. Solow, " A Complete Capital Model Involving Heterogeneous Capital Goods ", *Quarterly Journal of Economics*, **70** (1956), 537-662.

As the present section shows, if $f(k)$ is characterized by fixed-coefficients, marginal productivity is undefined at $k^\dagger$ and there is both room for and need for a short-run *alternative* theory of distribution.

Suppose now that many, but only a finite number of, alternative pages of mechanized blue prints are possible. If the alternatives are many and varied, putting r^* in different narrow intervals will cause the processes used to be different (and will usually involve differences in the calculated aggregate ratio of the value of capital to output). Any particular short-run r can prevail only if the proper plentitude and scarcity of diverse physical capital goods is available per capita. Given reasonable, but not perfect foresight, section's (1)′, in which $k^\dagger$ represents the minimum capital-output ratio, beyond which output expands not at all and k is technically redundant and before which labour is technically redundant. When output *per capita* is given by $y = \text{Min}(k/k^\dagger, 1)$, the capital-output ratio equals $k^\dagger$ for $k \leqq k^\dagger$ and equals k itself for $k > k^\dagger$.

The Kaldorian analysis concentrates on the case where k is at $k^\dagger$, neither factor being technically redundant. It does not seem to tell us what $r = \phi(k)$ is away from $k^\dagger$, and hence we must work out the conditions under which the system could stay at $k^\dagger$.

As both Kaldor and Pasinetti explicitly insist, the theory has a need to stipulate the inequality

$$s_w < nk^\dagger < s_c. \qquad \text{...(25)}$$

We shall now *deduce* this condition as a requirement for the system's being capable of staying at $K/L = k = k^\dagger$. Our treatment is different in deducing (25) from dynamic stability analysis of the differential equation of growth; but more important, it differs in that Kaldorians seem to think (25) must hold lest some factor share become negative or lest some regime of inflation or of unemployment become chronic. For us (25) must hold only if the system is to be capable of staying at $k^\dagger$; if (25) is violated, the system merely moves, in the manner already studied in section X and analyzed in more detail in the Appendix, toward $k = 0$ or toward technically redundant $k > k^\dagger$.

investors will start new physical capital goods and abandon old ones at a rate determined by the over-all (s_w, s_c) ratios and the inherited composition of physical capital goods. The result can be an approach toward a new golden age, with (the vector of) " capital scarcity and plenty " thought of as determining at each time the steadily evolving profile of own rates of interest, with no great surprises occurring but with some people experiencing good luck and some bad, and with most prudently remaking their plans as new experience warrants. In moving from a higher to a lower interest rate, any new golden age we come to is assuredly characterized by definitely higher real wages. And the lower interest rate does reflect a lower trade-off between consumption today and consumption next year. (Or, more precisely, one gets not more than $1+r$ of next-period C for sacrifice of each current C; and one gives up not less than $1+r$ of next-period C to get one more current C; the more varied and numerous the pattern of alternative blueprints, the more these inequalities narrow down toward the Fisher equality $|\partial C_{t+1}/\partial C_t| = 1+r$.) Because multiplicity of equilibria is possible, and " Wicksell effects " of market revaluation of capital can take place in any direction, one cannot be sure that a lower r^∞ always corresponds to a higher ratio of the value of capital to putput or even to a higher plateau of maintainable total consumption for a fixed population; in possible, so-called perverse cases, society might move from a high to a low interest rate state without having to abstain from current consumption goods, instead actually being splashed with a transient increase in consumption. Empirically, one expects the blueprints of technology to be such that at a lower interest rate, not only are wages higher because of less discounting of gross productivities, but also because the size of the social pie has gone up, and along with it the size of labour's undiscounted " productivity "; but exceptions to this are logically possible.

Given the above version of a useful pragmatic model, let us examine the Cambridge view expressed well by R. F. Kahn (op. cit.). He says: " Thrift is important, however, because it determines the real wage-rate in a Golden Age" (p. 151). We would say, " In our model certainly, and in most realistic models we suspect with great probability, past abstention from consumption was necessary to build up the stock of capital goods approriate to the high-real-wage-rate Golden Age; and continued abstention from utilizing resources for additional consumption goods is needed to replace and maintain the *per capita* supply of capital goods of that Golden Age. In our model, with interest low, real wages are two ways higher: output $f(k)$ is higher and, for each *same* level of k, less r means a lower subtraction of profit from $f(k)$ to get w; and, of course, the effects of K/L on absolute and relative profit shares depend on the (generalized) ordinary elasticity and on the (generalized) elasticity of substitution for k." Kahn says (p. 153): " If two different Golden Ages are compared, with the same saving coefficients but different rates of growth, the higher rate of growth is associated with the higher rate of profit. This higher rate of profit is to be attributed to the higher rate of growth of capital rather than the other way around." We say: " The unobjectionable first sentence does not logically (or empirically) imply the second. In our model the higher rate of profit persists in the higher-rate-of-growth regime because the stipulated limited thrift will, when L grows so fast, provide capital formation that will preserve only a low and scarce K/L ratio (or *per capita* vector of heterogeneous capital goods); and the resulting capital scarcity " explains " the high r. This is true not only in the simple model (1)-(4), but also in many models with tens-of-thousands of heterogeneous physical capital goods and technologies and with a large number of alternative blueprints. (Again, there are exceptions. And if Kahn and we drop the assumption that s_w is small, we can for some such exceptional cases find that a rise in the rate of population growth ends us up, for a long time or even permanently, with a lower rather than higher interest rate. This odd effect is inconsistent with simple neoclassical parables but not with the general market conditions of neo-neoclassical models.) The cited Champernowne article, precisely because it is mathematical, shows that some of what is considered different about Cambridge or macroeconomic theories of distribution is compatible with complete models of general-equilibrium pricing that we would espouse.

Begin by supposing $s_w > nk^\dagger$. There are two cases of this: $s_w > s_c$ or $s_w < s_c$. In the first, we know from previous discussion that our Dual analysis must hold and necessarily $k_c \to 0$ with $k_w \to k$. With capitalists' saving being ignorable, the familiar Harrod-Domar-Solow formula for a golden-age becomes applicable, namely $s/n = (k/y)$, where s and s_w are now the same. Since $s_w/n > k^\dagger$ by hypothesis, the equilibrium capital-output ratio must obviously end up at $k > k^\dagger$. So $k^\dagger$ is not viable.

In the second case, we have $s_c \geqq s_w > nk^\dagger$, and the ultimate saving ratio s, averaged over both classes, must satisfy: $s > nk^\dagger$, and $(k/y) = s/n > k^\dagger$ must be the equilibrium capital-output ratio. Again $k^\dagger$ is not viable.

To complete our derivation of (25), examine the remaining case where

$$\text{Max}\,(s_c, s_w) < nk^\dagger.$$

Any average of (s_c, s_w) must satisfy $s < nk^\dagger$. Therefore, we find for all K/L that $\dot{K}/K < n$ and $k \to 0$. Proof:

$$\frac{\dot{K}}{K} = s\left(\frac{Y}{K}\right) < nk^\dagger\left(\frac{Y}{K}\right) = nk^\dagger\,\text{Min}\left(\frac{1}{k^\dagger}, \frac{1}{k}\right) = n\,\text{Min}\left(1, \frac{k^\dagger}{k}\right) \leqq n.$$

Hence, $k^\dagger$ is again not viable.

By examining all cases, we have shown that unless (25) is satisfied $k^\dagger$ will not be viable. Regardless of the distribution of income α, either saving will be insufficient to widen capital in balance with L growth, or it will be so great as to deepen capital beyond the knife-edge $k^\dagger$. If (25) is satisfied, there is still no guarantee that $k^\dagger$ will be maintained and certainly no guarantee that from $k \neq k^\dagger$ the system will move in a stable way to $k \to k^\dagger$.

If (25) is satisfied and if we start the system with values of k_c and k_w that add up to $k^\dagger$ (and if, as will be soon discussed below, k_c is not too small), there will exist one distribution of income $\alpha^\dagger$ and concomitant $r^\dagger$—call it the Kaldor-Pasinetti one—which will make $\dot{k} = 0$ and hence maintain k momentarily at $k^\dagger$. While k remains at $k^\dagger$, its composition between k_c and k_w will generally be changed at the next moment; but if $\alpha^\dagger$ is always kept adjusted to the resulting $(k_c^\dagger, k_w^\dagger = k^\dagger - k_c^\dagger)$ so as to keep $\dot{k} \equiv 0$, we shall prove that $\alpha^\dagger \to \alpha^{\dagger\dagger}$, a unique Cambridge distribution of income for which $(\dot{k}_c, \dot{k}_w, \dot{k}) = (0, 0, 0)$ and at which $r^\dagger \to r^{\dagger\dagger} = n/s_c$ in good Pasinetti fashion. Just why any actual economic system should go to $k^\dagger$, and more importantly, being there should neatly settle into the $(\alpha^\dagger, r^\dagger)$ configuration is a topic that appears not to have been adequately discussed in the Kaldorian literature. And it is not discussed here. We merely state the conditions for the existence of $\alpha^\dagger$ and prove the asymptotic stability of $\alpha^{\dagger\dagger}$ if $\alpha^\dagger$ somehow is made always to prevail.

Theorem of Kaldor-Pasinetti type. If

$$C + \dot{K} = \text{Min}\left(\frac{K}{k^\dagger}, L\right) = L\,\text{Min}\left(\frac{k}{k^\dagger}, 1\right),$$

$$s_w < nk^{\dagger\dagger} < s_c$$

$$k = k^\dagger = k_c^\dagger + k_w^\dagger, \quad k_c^\dagger \geqq \frac{nk^\dagger - s_w}{s_c - s_w}\,k^\dagger, \qquad \ldots(25')$$

then

(i) there will exist an $(r^\dagger, \alpha^\dagger)$, given below, that will keep $\dot{k}$ in (5″) of Section V zero

$$\dot{k} = \left(\frac{s_w}{n} - k^\dagger\right)n + r^\dagger k_c^\dagger (s_c - s_w) = 0, \text{ or}$$

$$r^\dagger = \frac{nk^\dagger - s_w}{(s_c - s_w)k_c^\dagger}, \quad \alpha^\dagger = \frac{r^\dagger k^\dagger}{y^\dagger} = \frac{nk^\dagger - s_w}{(s_c - s_w)}\frac{k^\dagger}{k_c^\dagger} \qquad \ldots(26)$$

(ii) there will exist a golden age $(r^{\dagger\dagger}, \alpha^{\dagger\dagger})$

$$r^{\dagger\dagger} = \frac{n}{s_c}, \quad \alpha^{\dagger\dagger} = \frac{r^{\dagger\dagger}k^{\dagger}}{y^{\dagger}} = \frac{n}{s_c}k^{\dagger} \qquad \text{...(27)}$$

with $(k_c^{\dagger\dagger}, k_w^{\dagger\dagger})$ satisfying (26) and (27) and

$$k_c^{\dagger\dagger} = \frac{nk^{\dagger}-s_w}{(s_c-s_w)r^{\dagger\dagger}} = \frac{(nk^{\dagger}-s_w)s_c}{(s_c-s_w)n}, \; k_w^{\dagger\dagger} = k^{\dagger}-k_c^{\dagger\dagger} \qquad \text{...(27')}$$

and such that

$$\lim_{t\to\infty} (k_c^{\dagger}, r^{\dagger}, \alpha^{\dagger}) = (k_c^{\dagger\dagger}, r^{\dagger\dagger}, \alpha^{\dagger\dagger}) \qquad \text{...(28)}$$

no matter what the initial condition $k_c^{\dagger}$, provided it is large enough to satisfy (25′) ensuring that $\alpha \leqq 1$ in (26).

The proof of (i) follows directly from the indicated equation (5″). Its common-sense meaning is clear: by making r and α big enough or small enough, we generate more or less saving and for one critical level $r^{\dagger}$ and $\alpha^{\dagger}$ corresponding to the given capital ownership, we can make $\dot{K}/K$ exactly equal to n and $\dot{k} = 0$. Actually, from the first equation of (5), which gives $\dot{k}_c/k_c$ in terms of r, and from (26)'s equation for $r^{\dagger}$, we derive the following differential equation for $\dot{k}_c^{\dagger}$ in terms of $k_c^{\dagger}$ alone

$$\dot{k}_c^{\dagger} = \frac{nk^{\dagger}-s_w}{s_c-s_w}s_c - nk_c^{\dagger}. \qquad \text{...(29)}$$

Equation (29) enables us to prove (ii) of the theorem. Evidently at $\dot{k}_c^{\dagger} = 0$, we get the unique stationary point, which can be labelled

$$k_c^{\dagger\dagger} = \frac{nk^{\dagger}-s_w}{s_c-s_w}\frac{s_c}{n}$$

as in (27′). Since the general solution to (29) can be written in the form

$$k_c^{\dagger}(t) = k_c^{\dagger\dagger} + (k_c^{\dagger}(0) - k_c^{\dagger\dagger})e^{-nt}$$

the fact that $n>0$ ensures that $e^{-nt}\to 0$ and $k_c^{\dagger}\to k_c^{\dagger\dagger}$. Again the common sense of the proof is intuitive. If the division of $k^{\dagger}$ among capitalists and workers originally is lopsided in favour of the thrifty capitalists, $r^{\dagger}$ and $\alpha^{\dagger}$ will have to be initially low if excessive saving is to be avoided. But with $\alpha^{\dagger}$ so low, K_c grows at a lower rate than K_w, thereby tending to make $k_c^{\dagger}$ less lopsided than at the beginning. Thus, the correction continues until $k_c^{\dagger}$ has approached the critical $k_c^{\dagger\dagger}$ level at which $\dot{K}_c/K_c$ and $\dot{K}_w/K_w$, as well as $\dot{K}/K$, all grow at rate n.

All the above Kaldor-Pasinetti relations have a relevance, meaning, and definition only at the knife's-edge $k = k^{\dagger}$. For $k \neq k^{\dagger}$, they are vacuous. The marginal productivity theory of perfect competition is exactly the opposite: for $k \neq k^{\dagger}$, marginal productivities are well-defined—with capital redundant and $r = 0 = \alpha(k)$ if $k>k^{\dagger}$, and with labour redundant and $w = 0 = 1-\alpha(k)$ if $k<k^{\dagger}$. But at the knife-edge $k = k^{\dagger}$ where the production function has a sharp edge or corner, marginal-productivity derivatives are undefined and the simple neoclassical theory becomes vacuous. We might therefore try to perform a marriage of complementary opposites, defining $\phi(k)$ by marginal productivity as 0 for $k>k^{\dagger}$ and as $1/k^{\dagger}$ for $k<k^{\dagger}$, but defining ϕ at $k^{\dagger}$ by the $r^{\dagger}$ formula of (26).

What makes the marriage of some interest is the fact that if $\phi(k)$ is defined away from $k^{\dagger}$ by the above marginal productivity relation, from any initial $k \neq k^{\dagger}$ the system will move to $k^{\dagger}$ in finite time regardless of how $\phi(k^{\dagger})$ itself is to be defined—a property previously lacking in the Kaldorian models. The proof is straightforward. Suppose $k>k^{\dagger}$ initially. Then $\alpha(k) = 0$ and s_w alone counts. Then always

$$\dot{K}/K = (s_w)Y/K < nk^{\dagger}\text{ Min }(1/k^{\dagger}, 1/k) = nk^{\dagger}/k < n$$

and hence $k \to k^{\dagger}$ from above. Now suppose $k < k^{\dagger}$ initially. Then $\alpha(k) = 1$ and

$$\dot{K}/K = (s_c)Y/K > (nk^{\dagger}) \operatorname{Min}(1/k^{\dagger}, 1/k) = n,$$

and hence $k \to k^{\dagger}$ from below.[1]

Once we are at $k^{\dagger}$, if $k_c^{\dagger}$ is not so small as to be incompatible with $\alpha^{\dagger} \leqq 1$, and if r is put at (26)'s $r^{\dagger}$ level, our theorem guarantees an ultimate approach to the golden age of $r^{\dagger\dagger} = n/s_c$.

But we must hasten to add that whatever the value of the above model as an exercise, its economic relevance is in our view very dubious. Our scepticism applies to that portion of the path where $k \neq k^{\dagger}$ as well as to the Kaldor-Pasinetti regime. The 100 per cent share of capital that corresponds to $k < k^{\dagger}$ is as devoid of realism as is the assumption that L will continue to rise even at an imputed wage of zero. And the zero rate of return to capital associated with $k > k^{\dagger}$ poses equally formidable questions concerning the problem of maintaining adequate aggregate demand. Within the Kaldor-Pasinetti regime at $k = k^{\dagger}$, there arises in our view equally serious questions about the relevance of the model. For it is one thing to write down equations like (26) to exhibit the values of r and α for which k will remain at $k^{\dagger}$ but quite a different one *to exhibit the behavioural mechanism which will ensure that the actual distribution of income will be precisely that required by* (26). In our view none of the macroeconomic-distribution writers have thus far been able to provide a convincing formulation of the required mechanism.[2]

The perfectly competitive market gives supply and demand curves at $k = k^{\dagger}$ that are coincidental vertical lines over the range of wage share from 0 to 100 per cent of national income. If oligopoly elements are to be introduced, or national collective bargaining, a wide variety of outcomes are possible. A theory of the corporate state or of great good luck could arbitrarily posit that national bargaining results in a distribution which makes the weighted average of full-employment full-capacity thriftiness just enough to finance the widening of capital needed to match the natural rate of growth of the effective labour force. That would indeed be a theory of serendipity.[3] If one posits employers who make contracts to hire labour at fixed money wages in the short run and who earn profits as a short-run residual, one must decide what theory of price flexibility and price inflexibility is supposed to prevail in each run of time. The notion of a floating profit or interest rate r, which floats (in a fixed-coefficient model) as a result of flexible price/wage margins to produce just that distribution of income that will induce warranted-saving-growth-rates exactly equal to the system's natural rate of growth set by population and Harrod-neutral

[1] Will any other $\phi(k)$ function lead to $k \to k^{\dagger}$? Yes, there are many such functions that will suffice once (25) is satisfied with particular strength. But if (25) can hold with *any* strength or weakness of the inequalities, then only the competitive $\phi(k)$ can be counted on to lead *always* to $k \to k^{\dagger}$.

[2] One of us has presented a logically complete (but empirically bizarre) short-run Kaldorian full-employment model—the only logically complete model known to us that deduces full employment by a well-defined dynamic process—in R. Leckachman, *Keynes General Theory*: *Report of Three Decades* (Macmillan, London, 1964), final chapter by P. A. Samuelson, particularly the equations on p. 344. The present, long-run version of a Cambridge-like system was given by the latter at Berkeley in January 1964.

[3] " All this concerns the analysis of relative shares under conditions of full employment. But full employment is a postulate, not a result of the theory." (J. Robinson, *Collected Economic Papers*, Vol. II (Blackwell, Oxford, 1962), p. 157.) Historically, market economies like the U.S. and U.K. have not been uniformly at full employment. Their unemployment has oscillated, averaging out to where it has averaged out. But this is to say almost nothing. If one goes on to argue—as Marshall, Pigou and Dr Kaldor have occasionally argued—that average unemployment has been remarkably small and remarkably trendless, the question arises as to what one should mean by " small " and by " remarkable ". That hurdle somehow bypassed, how tempted is one to infer an efficacious full-employment mechanism operating within the system? And is it supposed to be classical price, money-wage, real-wage flexibility? Is it Pigou-Patinkin effects? Is it Tobin-Solow central-bank interest-rate flexibility? Is it New-Deal fiscal policy that negates or offsets fixed (s_c, s_w) coefficients? Is it God's grace? Or is it teleological shifts in the distribution of income between thrifty and thriftless that, in some run of time, assures a stylized performance of high employment with reasonable price stability? If you can believe the latter, you can—as the Duke of Wellington once said—believe anything.

technical change, cannot be left to depend on an Invisible Hand never seen on land or sea, in Timbuktu or in either Cambridge.[1]

We conclude, therefore, that the writers on macroeconomic theories of distribution have not succeeded in reconciling the relative smooth functioning of behaviour and of share imputation in observed economic systems with the properties to be expected from a fixed-coefficient model of technology. This conclusion only reinforces the scepticism generated by many other types of evidence about the adequacy of the fixed-coefficient model as a useful first approximation to modern technology. Within each process fixity of proportions may well be realistic, but there are thousands of different processes in any modern society. At worst the steps within which α is an indeterminate vertical locus are minute, being nothing like the 0 to 100 per cent of the fixed-coefficient model. In a realistic model, we suspect that relations like (27) above would only determine the higher decimal places of relative shares, a result of limited interest in view of the stochastic fluctuations of such shares anyway.[2]

XII. A NEOCLASSICAL KALDORIAN CASE

However, if one had the empirical hunch that the elasticity of substitution is *almost* zero about some critical level $k^{\dagger}$, we can approximate the Kaldor results by the following well-behaved neoclassical limiting sequence. Let $f(k)$ not have a corner at $k^{\dagger}$ but rather greater and greater $f''(k)$ curvature there, so that $f'(k)$ is well-defined but becomes very steep near $k^{\dagger}$. Then, for a wide range of n, the system will indeed settle down near $k^{\dagger}$ (as can be proved by our methods applied to well-behaved neoclassical systems) provided only that a restriction like (25) holds. If in Fig. 4 we draw the marginal curve of Fig. 2 as if it is is almost vertical near its intersection with n/s_c close to $k^{\dagger}$, then the long-run comparative statical properties of the system will be—to coin a phrase—almost-Kaldorian. That is $\alpha^{\infty} = (n/s_c)(k/y) \doteq (n/s_c)k^{\dagger}$. Because k/y changes little from $k^{\dagger}$, the elasticity of α with respect to s_c is almost -1, which is all that Kaldor's Widow's Cruse boils down to. Note that this Kaldorian conclusion comes completely from marginal productivity relations, once (25) relations hold, there being no need for a genuine alternative theory of distribution! The stylized facts of mixed capitalism do not, alas, seem to us consonant with such violent induced changes in relative shares.

XIII. FINAL COMMENTS

We have shown that a one-sector neoclassical model with two classes of savers, a class who *forever* save a constant proportion s_c of their income that comes wholly from profits, and a class who *forever* save a constant proportion s_w of their income from wages

[1] Actually, in a heterogeneous capital model, where K stands for a vector of diverse capital goods and processes—alpha, beta, gamma . . . machines, inventories, etc.—there is no technically given $k^{\dagger}$ or aggregative capital-output ratio. Thus, in the artificial but instructive Surrogate capital model of P. A. Samuelson, "Parable and Realism in Capital Theory: The Surrogate Production Function," *Review of Economic Studies*, **29** (1962), 193-206, where an aggregative magnitude $J = K$ enters in the long-run production function $Y = F(J, L)$, there will be a different J/L that plays the role of $k^{\dagger}$ for each different range of long-run profit or interest rate—r or R. Except for indeterminacy in narrow ranges between the critical R levels at which one machine or another becomes optimal, the neoclassical equations stemming from (5) give a better approximation to the properties of this Neumann-Robinson-Sraffa-Dosso model than does the implicit theorizing of (26) and what follows in this present section.

In a competitive fixed-coefficient model, a relation like $f'(k^*) = n/s_c$ or $R(k^*) = n/s_c$ is lacking to permit us to solve for k^* and all the equilibrium values of the system. Formally, the missing equation $k^* = R^{-1}(n/s_c)$ is provided by setting long-run $k^* = k^{\dagger}$, the minimum technical capital-output (and, with our convention of setting $\lambda = 1$ in Min $(K/k^{\dagger}, L/\lambda)$, $k^{\dagger}$ is also the capital-labour ratio at full capacity).

We shall not here explore the possibility of a lagged-adjustment mechanism of α_k to its competitive $f'(k)$ level when $k = k^{\dagger}$ and to its $\alpha_k^{\dagger}$ serendipity level as defined earlier when $k = k^{\dagger}$. Call these norms $\alpha_k = N(k_c, k_w)$. Then $\dot{\alpha}_k = \beta[N(k_c, k_w) - \alpha_k]$ will, for proper positive time-constant β, generate together with (26) a sequence with the "Cambridge" property $\alpha(t) \to \alpha^{\dagger\dagger}$, as defined above.

[2] Refer back to footnote 1, p. 290.

and profits—will, if full-employment saving is always exactly matched by investment, approach a golden-age equilibrium state of exponential growth at n, the natural rate of population growth (augmented by Harrod-neutral technical change).[1] We have also shown that this result can be generalized to some non-neoclassical models. Provided the profit rate is a single-valued, declining function of the capital/labour ratio (and even if competitive imputations of marginal productivity do not define this function), there also exists a stable golden age.[2] And we have deduced, on the basis of the special inequalities and equalities posited by Kaldorian distribution theory, the asymptotic stability of the minimum technical ratio of the value of capital to output in one particular fixed coefficient

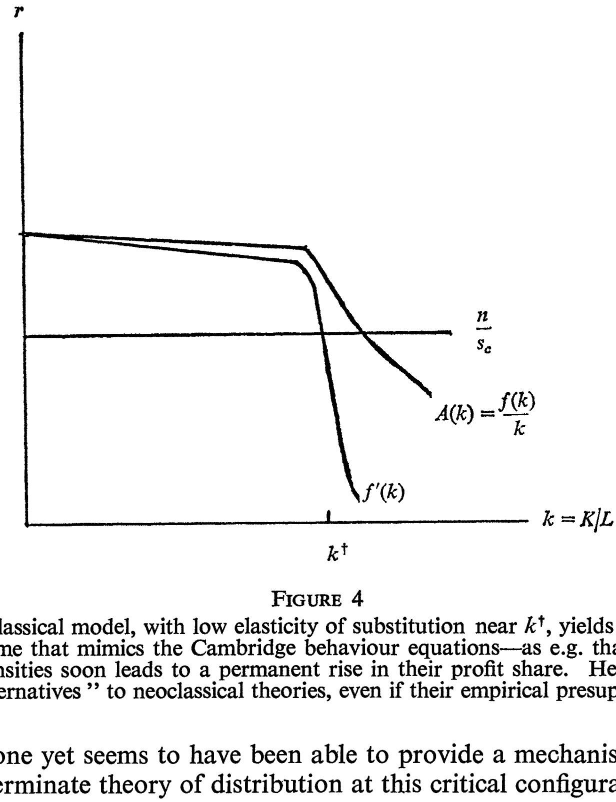

FIGURE 4

A smooth neoclassical model, with low elasticity of substitution near $k^\dagger$, yields marginal-productivity distribution of income that mimics the Cambridge behaviour equations—as e.g. that a rise in capitalists' consumption propensities soon leads to a permanent rise in their profit share. Hence, such theories are not necessarily " alternatives " to neoclassical theories, even if their empirical presuppositions are realistic.

model. But no one yet seems to have been able to provide a mechanism from which one can *deduce* a determinate theory of distribution at this critical configuration. If, however, it is arbitrarily postulated that the distribution of income there will somehow become precisely such as to keep total capital growing in balance with labour, we have proved the stability of the asymptotic Pasinetti state with its unique distribution of income.

Our analysis confirms the beautiful asymptotic theorem of Dr Pasinetti for a limited range where s_w/s_c is small enough. And it provides a completely symmetrical Dual theorem —where average-product-of-capital $= n/s_w$ replaces his net-yield-of-capital $= n/s_c$, etc.— for the (empirically quite interesting) range where s_w/s_c is greater than $\alpha(k^*)$, profits' share in national income. We have demonstrated that the applicability of either of these regimes depends on $s_w \gtreqless \alpha^\infty s_c$ (where α^∞ is the golden-age share of profits, by whatever theory determined), a criterion that has nought[3] to do with the issue of whether smooth neo-

1 The qualification concerning a Pasinetti limit cycle of Section 6 is in order here.

2 When r does not, for some K/L, run the full gamut from infinity to zero, the Appendix shows that the only asymptotic states may involve $k \to 0$, or $r \to$minimum r with k perhaps going to infinity. These " pathological states " are shown to be also stable.

3 Recall, though, last section's demonstration that the Kaldorian distribution theory does presuppose the restriction $s_w < nk^\dagger < s_c$, which *a fortiori* puts us in the Pasinetti regime $s_w < \alpha^\infty s_c \leqq s_c$.

classical or fixed-coefficient neo-neoclassical technologies are involved, or of whether imperfect competition modifies competitive marginal productivities.[1] Dr Pasinetti's results and our extensions of them have a generality that can encompass valid theories in Cambridge, Massachusetts, Cambridge, Wisconsin, or any other Cambridge. On the other hand, we have pointed out some of the real problems created if only a single fixed-coefficient technology (or narrow spectrum of technologies) is assumed.

Many realistic complications ought to be added to our analysis: the introduction of uncertainty, for example, and of heterogeneous capital-goods activities of the modern programming or Sraffa type. Some of these complications are quite easily handled by modern methods. Some offer intrinsic difficulties.

But quite aside from such modification of the model, we feel it necessary to conclude with a warning about the extremely unrealistic nature of some of our crucial assumptions. Our warning is not merely directed against oversimplifications, like exponential growth of the equivalent labour force, that are unlikely to be literally true. With a grain of salt, we cheerfully make such heroic abstractions as a first approximation—making sure to determine later whether the results depend critically upon the *exactitude* of the abstract axioms.

We are much more uneasy with the assumption of "permanent" classes of pure-profit and mixed-income receivers with given and unchanging saving propensities on which all of our theorems—Pasinetti's as well as ours—depend critically. This assumption completely disregards the life cycle and its effect on saving and working behaviour. In the first place with a large portion of saving known to occur in some phases of the life cycle in order to finance dissaving in other phases, it is unrealistic to posit values for (s_c, s_w) which are independent of n.[2] This shortcoming is probably not too serious and could be handled without changing our results drastically.

But the assumptions of *permanent* classes of income receivers raise much more serious questions. Even if we wave aside the difficulty of identifying a class whose sole source of income is income from capital, there is no reason to suppose that a person who belongs to that class at some point of his life must have always belonged to it and will continue to do so indefinitely. In a modern industrial society the capitalist's class is not a hereditary caste: its membership at any point of time is far from limited to people who were born into it by virtue of inherited wealth. This becomes especially clear when we recall that by our definition, the capitalist class would include retired households living off their capital. But even the assumption that class membership and saving propensity do not change during one person's lifetime is not enough for our purposes. Since people do not live forever, one would have to extend the asumption to one's heirs, and their heirs, and so on, until Kingdom-come—both before golden ages are reached and forever afterwards. For, the moment that one admits transfers of wealth between classes, either through living persons switching class membership or by virtue of death, *it is no longer true that* $\dot{K}_c$ *and* $\dot{K}_w$ *are equal to the rate of saving of the respective classes* and therefore differential equations of the type (3) that we and Dr Pasinetti have postulated are no longer satisfied. That being the case, simple experiments will show that results quite different from our nice theorems can result.

1 Even without our $\phi(k)$ function, a necessary criterion for a Pasinetti regime is $s_w<\alpha^*s_c$ and a necessary criterion for an anti-Pasinetti regime is $s_w \geqq \alpha^{**}s_c$. If $\phi(k)$ is posited, with $\phi'(k)<0$, these are equivalent and only one regime is possible. Actually, this same conclusion would follow if r were a monotone-decreasing function of the value of capital (measured in consumption goods). If no restrictions on r are placed, one can encounter alternate Pasinetti and Dual golden ages, some of which might be locally stable.

2 See e.g., Modigliani, "The Life Cycle Hypothesis of Saving," paper presented at the First International Meeting of the Econometric Society, Rome, Italy, September 1965 (mimeo); and "The Life Cycle Hypothesis of Saving, the Demand for Wealth and the Supply of Capital," *Social Research*, 33 (1966), 160-217; and the bibliography cited therein. Cf. also the following important contribution, which has appeared since that bibliography was completed: J. E. Meade, "Life-Cycle Savings, Inheritance, and Economic Growth," *Review of Economic Studies* 33 (1966), 61-78.

Therefore, we conclude with the caution: Beware.

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APPENDIX: PATHOLOGICAL CASES OF DIVERGENCE

In this Appendix we analyze certain " pathological " cases which keep coming up in the literature and seem nowhere to have been treated comprehensively. The two different pathologies refer to the cases where $k \to 0$ and where $k \to \infty$, cases which were excluded by our assumptions in footnote 3, p. 269. We now relax those assumptions.

First, consider the case of what can happen when $k = K/L \to 0$. If average product of capital $A(k)$ and marginal product $f'(k)$ both approach infinity as $k \to 0$, as is suggested by Fig. 2's curves, our general theorem needs no modifications. The horizontal lines of Fig. 2 do intersect the relevant curves and $k \to k^{\infty} > 0$.

But if one or both of these approach a finite level as $k \to 0$, as in the accompanying Fig. 5 redrawing of Fig. 2, then for n large enough (or both saving propensities small enough), the Pasinetti equation $f'(k^*) = n/s_c$ will have no root at all and will become an inequality. (The n/s_c horizontal now lies above the marginal curve everywhere.) If s_w is positive and $A(k) \to A(0) = \infty$, no modification of our theorems is required: for even if $n/s_c > \text{Max}\, r = r_m$, $A(k^{**}) = n/s_w$ can still be satisfied for $k^{**} > 0$; the n/s_w horizontal intersects the average curve in Fig. 5, and *all* the conclusions of our Dual theorem become fully applicable. (The case where $s_w = 0$ can best be handled as a footnote to the next paragraph's case.)

Suppose now that as $k \to 0$, $A(k) \to \text{Max}\, A = A(0) = A_m < \infty$. Then necessarily from the familiar relationship of identical finite intercepts for marginal and average curves, $r = f'(k) \to f(0) = r_m = A_m$. With n sufficiently small compared to $\text{Max}\,(s_c, s_w)$ no modification in our theorems are needed, since $k \to 0$ will not then occur. But when n becomes sufficiently large, neither equation $f'(k^*) = n/s_c$ or $A(k^{**}) = n/s_w$ can be satisfied; both horizontals in Fig. 5 will now be above the average and marginal curves; and then, of course, $\dot{k}$ is always negative and pushes k back to zero. The common sense reason for this is plain. If effective L now grows very fast (and/or if saving propensities are very low), the system will not be able to have enough capital formation to widen capital in balance with effective labour. The amount of capital per efficiency unit of labour steadily retrogresses, as we shall see, sometimes with dire Malthusian consequences for the hourly wage rate. Let us consider in detail the different cases.

(i) $n > \text{Max}\,(s_c, s_w)A_m < \infty$. Then (3) and (5) imply

$$(k_c, k_w, k, y;\ r, A, \alpha, w) = (0, 0, 0, 0;\ A_m, A_m, 1, 0)$$

$$\lim_{t \to \infty} \left\{ \left[\frac{\dot{K}_c}{K_c}, \frac{\dot{K}_w}{K_w}, \frac{\dot{K}}{K}, \frac{\dot{Y}}{Y} \right] \right. = A_m[s_c, s_w, \text{Max}\,(s_c, s_w), \text{Max}\,(s_c, s_w)]$$

$$\frac{K_c}{K} \to 1 \text{ if } s_c > s_w; \quad \frac{K_w}{K} \to 1 \text{ if } s_w > s_c;$$

If $s_w = s_c$, $\left[\dfrac{K_c}{K}, \dfrac{K_w}{K} \right]$ are indeterminate, being dependent on initial conditions.

(ii) $n = \text{Max}\,(s_c, s_w)A_m < \infty$, and one of the horizontals intersects the curves at the A_m intercept level itself.

The same conclusions hold as above, unless $A(k) = A_m$ holds for all k in the range $0 \leqq k \leqq b>0$. In that case k^∞ ends in that range rather than necessarily at zero, just where being dependent on initial conditions; if also $s_c = s_w$, the allocation of k^∞ between (k_c^∞, k_w^∞) will also depend on initial conditions. (If any $K \equiv 0, \dot{K}/K = 0/0$ and by harmless convention can be made to agree with all statements made.)

Since no modifications are needed when $n<\text{Max}\,(s_w, s_c)A_m$, we can complete our discussion of the pathology where $k \to 0$ with a final observation. The fact that income per efficiency unit, Y/L, goes to zero in the limit does not necessarily mean that genuine

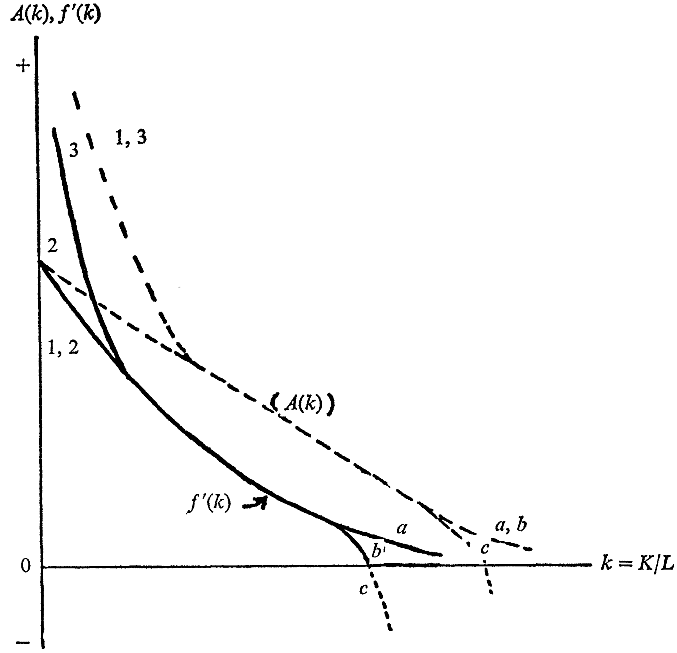

FIGURE 5

Branches 1, 2 and 3 represent variant pathologies as $k \to 0$; branches a, b, c and d (not shown) refer to the case where $k \to \infty$.

per capita income goes to zero. Capital may fail to widen relative to the enhanced supply of effective labour generated by the Harrod-neutral technical change coefficient n'', while still being capable of keeping up with the n' rate of demographic change. The condition for $K(t)$ to grow faster than the work force, $L_0e^{n't}$—even though $n>\text{Max}\,(s_w, s_c)A_mk \to 0$—turns out to be $s_wA_m>n' = n-n''$. If this is the case, the economy is becoming more prosperous on a true *per capita* basis even though $k \to 0$.

If capital is not growing as fast as the natural labour force, it is a foregone conclusion that the ultimately zero wage per efficiency unit must also imply an ultimately zero wage rate per natural unit of labour. But, what happens when $n'<s_wA_m$ and K does grow faster than natural L? There is then the possibility that the wage rate per natural unit of labour might rise toward infinity even though the wage rate per efficiency unit of labour goes to zero. Indeed one might at first intuition conjecture that this has to happen. But that would be a wrong guess. The stability analysis of (14) and (16) shows that $K(t)$ grows asymptotically like $e^{s_wA_mt}$, that $k(t)$ shrinks asymptotically like $e^{-(n-s_wA_m)t}$, and (surprise!) that $w(t)$ shrinks like $[k(t)]^2$ or like $e^{-2(n-s_wA_m)t}$ (this because at $k = 0$ the slope of w as a function of k turns out to be zero). Consequently the condition that the wage per natural L rise is that $w(t)e^{n''t} = e^{(n''-2n+2s_wA_m)t}$ have positive exponent, giving us in the end the

criterion $n''/2 \gtreqless s_w A_m - n'$ for the wage per natural worker to rise, stay the same or fall. The fact that for a sufficiently high rate of Harrod neutral technical change n'', the $>$ sign would hold in the above condition, may help to make logical sense of the Marxian foreboding of a progressive immiseration of the working class due to technical change rather than due to Malthusian biology. It should be noted that, in the case under consideration, so-called Harrod *neutral* technical change is actually *labour saving* in the Hicksian sense. It is a general truth that Harrod neutral technical change is Hicksian labour saving if the elasticity of substitution is less than one, and this elasticity is assuredly less than one in the neighbourhood where the average and the marginal product of capital approach a common finite intercept. Note also that what would be needed to help cure this Marxian kind of poverty—should it ever come about—is *more* rather than less effective thrift!

The possibility that K/L goes to infinity can now be analyzed. Suppose $n = 0$ and Min $(s_w, s_c) > 0$. Then it is obvious that positive saving and investment will go on indefinitely, and that K and k will grow indefinitely. Diminishing returns implies that the interest rate will go to its minimum r_M. (The right hand of Fig. 5 shows with branches a, b and c the possible patterns of r_M.) If some positive labour is needed for positive output, the minimum interest rate cannot be positive. If capital never has a negative productivity, $r_M = 0$. However, it is easy to imagine production processes where $f'(k) = 0$ for a finite k^*; on branch c, further k produces a positive gross quasi-rent, but not one large enough to equal depreciation, and hence produces a negative $f'(k)$. In this case, income drops with further k, until finally $A(k) = 0$ has a finite root k^{**} on its c branch. So long as $s_w > 0$, the system will end up in the Dual k^{**} equilibrium, with

$$k_c^\infty / k_w^\infty = 0, \dot{K}/K = \dot{K}_w/K_w \to 0, \text{ and } r \to f'(k^{**}) < 0:$$

the workers ultimately make profit losses equal to their saving, and enjoy less consumption than they get when $s_w = 0$. For with $n = 0$, $s_c > 0 = s_w$, we end in the Pasinetti state k^*, with $r \to 0$, wages$\to$100 per cent y = 100 per cent maximum Golden Rule y.

If $f'(k) > 0$ for $k < k^*$ but $f'(k) \equiv 0$ for $k > k^*$, $A(k)$ is always positive and for $n = 0 < s_w$, $\dot{k} > 0$ forever. However, $\dot{K}/K = \dot{K}_w/K_w \to 0$, $r \to 0$, $K_c/K \to 0$, and $w \to$Max w. We are in the Dual regime, but with $k^{**} = \infty$, so to speak. (If $s_w = 0 = s_c = n$, there is never anything to talk about: the system stays forever at initial (k, r, w) configuration. If $n = 0 = s_w < s_c$, we are back to the previous Pasinetti case whether or not $f'(k) < 0$ beyond k^*.)

We are left now with the case where $f'(k) > 0$ forever, and *a fortiori* $A(k) > f'(k) > 0$. If $f'(k) \to 0$ as $k \to \infty$, as on branch a, there are still two possibilities: $f(k) \to f(\infty) < \infty$, implying that $\alpha(k) \to \alpha(\infty) = 0$ (just as in the case where $f'(k^*) = 0$ for finite k^*). Again, if $s_w > 0$ at all, the Dual regime dominates with all that this implies. If $s_w = 0 < s_c$, the system approaches the infinite Pasinetti state, $k^* = \infty$, with $k_c/k \to 1$.

The second possibility is where $f(k) \to f(\infty) = \infty$. If $f'(k)/A(k) = \alpha(k)$ should approach a definite limit (and it need not), we are in the infinite Pasinetti or infinite Dual regime depending on whether $s_w \lesseqgtr \alpha(\infty)s_c$, just as in the finite case. In the Dual case, $k_c/k_w \to 0$ as usual. In the Pasinetti case, $k_c/k_w \to k_c^*/k_w^* > 0$, whose value can be computed by the happy circumstance that in the last equation of (6), the ratio k_c^*/k_w^* proves to be independent of n (except for n's influence on k^*, of course): it is given by $[s_c\alpha(k^*) - s_w]/s_w[1 - \alpha(k^*)]$, where here we take $\alpha(k^*) = \alpha(\infty)$. Paradoxically, although $(\dot{K}_c/K_c, \dot{K}_w/K_w) \to (0, 0)$, $[(\dot{K}_c/K_c)/(\dot{K}_w/K_w)] \to 0/0 = 1$! The stability of these limits follows from our differential equations: it can be checked by working out the Cobb-Douglas case, since $\alpha(k) \to \alpha(\infty)$, a definite limit, implies that the production function can be regarded as *asymptotically* Cobb-Douglas.

We are left with the v. Neumann case where zero (exogenous) labour is compatible with positive production, and where Min $r = r_M = A_M > 0$. (To keep the diagram from becoming too cluttered, we do not show this case on Fig. 5; but an example would be $f(k) = r_M k + ak$.) Now $\alpha(\infty) = 1$, and $[\dot{k}_c/k_c, \dot{k}_w/k_w, \dot{k}/k] \to r_M[s_c, s_w, \text{Max}(s_c, s_w)] > n$.

This all holds as much for small positive n as for zero n. With $\alpha(\infty) = 1$ necessarily, ratio $k_w/k_c \to 0$ whenever $s_c > s_w$, as is shown by the above formulas or by the general formula of the previous paragraph.

Whereas this v. Neumann case may be empirically uninteresting, it duplicates the complexity of the possible case of population decline, where $n<0$. Here again $k\to\infty$, whenever algebraic $n<r_M$.

We have not given the detailed, but straightforward, stability analysis underlying the many assertions of this technical section. A pathological example that embodies the branch 1 and the v. Neumann case, is given by $F(K, L) = \bar{r}K+\bar{w}L$. Here the fate of capitalists is independent of labour, $r(t) \equiv \bar{r}$, and $K(t) \equiv K(t_0)\exp(\bar{r}s_c)t$: for $n/s_c<\bar{r}$, $k_c\to\infty$; for $n/s_c = \bar{r}$, k_c stays at its initial value. If $s_w>s_c$, $K_c/K_w\to 0$. In every case, K itself grows at the fastest of the rates $(n, s_c\bar{r}, s_w\bar{r})$, so that $k\to 0$ is impossible. If $s_w<n/\bar{r}<s_c$, $K_w/K_c\to 0$, in agreement with (6)'s last equation because then $\alpha(\infty) = 1$.